Chapter Three
Relations on Sets

3.1 Cartesian Product

Definition 3.1.1. A set A is called

(i) finite set if A contains finite number of element, say n, and denote that by $|A| = n$. The symbol $|A|$ is called the **cardinality** of A,

(ii) infinite set if A contains infinite number of elements.

Definition 3.1.2. The **Cartesian product (or cross product)** of A and B, denoted by $A \times B$, is the set $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

(1) The elements (a,b) of $A \times B$ are ordered pairs, a is called the **first coordinate (component)** of (a,b) and b is called the **second coordinate (component)** of (a,b).

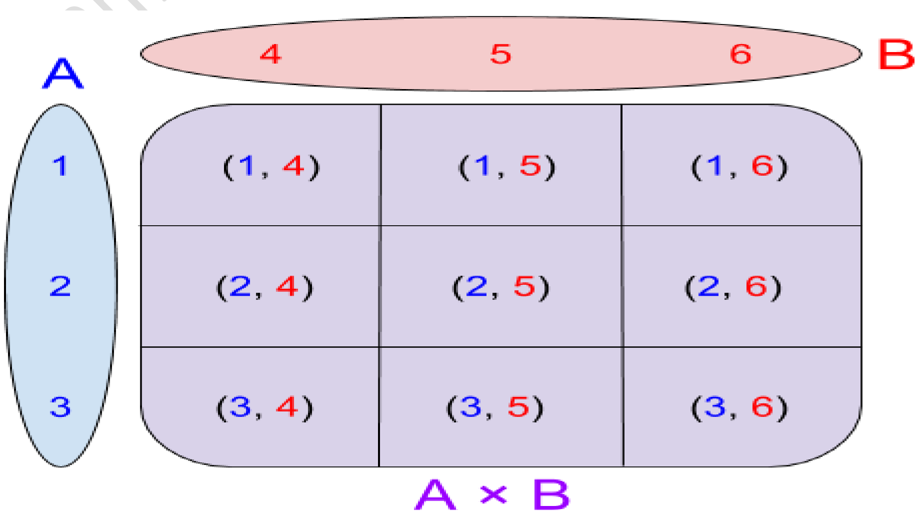
(2) For pairs $(a,b),(c,d)$ we have $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$.

(3) The n-fold product of sets $A_1, A_2, \dots, A_n$ is the set of n-tuples

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for all } 1 \le i \le n\}.$$

Example 3.1.3. Let $A = \{1,2,3\}$ and $B = \{4,5,6\}$.

(i) $A \times B = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$.



(ii) $B \times A = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\}$.

Remark 3.1.4.

(i) For any set A, we have $A \times \emptyset = \emptyset$ (and $\emptyset \times A = \emptyset$) since, if $(a,b) \in A \times \emptyset$, then $a \in A$ and $b \in \emptyset$, impossible.

(ii) If $|A| = n$ and $|B| = m$, then $|A \times B| = nm$. Also, A or B is infinite set then cross product $A \times B$ is infinite set.

(iii) Example 3.1.3 showed that $A \times B \neq B \times A$.

Theorem 3.1.5. For any sets A, B, C, D

(i) $A \times B = B \times A \Leftrightarrow A = B$,
(ii) if $A \subseteq B$, then $A \times C \subseteq B \times C$,
(iii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$,
(iv) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,
(v) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$,
(vi) $A \times (B - C) = (A \times B) - (A \times C)$.

Proof.

(i) The necessary condition. Let $A \times B = B \times A$. To prove $A = B$.

Let $x \in A \Longrightarrow (x,y) \in A \times B, \forall y \in B$. Def. of $\times$

$\Longrightarrow (x,y) \in B \times A$ By hypothesis

$\Leftrightarrow x \in B \wedge y \in A$ Def. of $\times$

(1) $\Longrightarrow x \in B \Longrightarrow A \subseteq B$ Def. of $\subseteq$

(2) By the same way we can prove that $B \subseteq A$.

Therefore, $A = B$ Inf(1),(2).

The sufficient condition. Let $A = B$. To prove $A \times B = B \times A$.

Since $A \times A = A \times A \Longrightarrow A \times B = B \times A$ By hypothesis.

(vi) $A \times (B - C) = (A \times B) - (A \times C)$.

$(x,y) \in A \times (B - C) \Leftrightarrow x \in A \wedge y \in (B - C)$ Def. of $\times$

$\Leftrightarrow x \in A \wedge (y \in B \wedge y \notin C)$ Def. of $-$

$\Leftrightarrow (x \in A) \wedge (x \in A) \wedge (y \in B \wedge y \notin C)$ Idempotent Law of $\wedge$

$\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C)$ Comut. and Assoc. Laws of $\wedge$

$\Leftrightarrow (x,y) \in (A \times B) \wedge (x,y) \notin (A \times C)$ Def. of $\times$

$\Leftrightarrow (x,y) \in (A \times B) - (A \times C)$ Def. of $-$