Definitions 3.2.6.
(i) R is an **equivalence relation** A, if R is reflexive, symmetric, and transitive. The set

$$[x] = \{y \in A: xRy\}$$

is called **equivalence class**.

(ii) R is a **partial order** on A(an **order** on A, or an **ordering** of A), if R is reflexive, antisymmetric, and transitive. We usually write $\leq$ for R, i.e.

$$x \leq y \text{ iff } xRy.$$

(iii) If R is a **partial order** on A, then the element $a \in A$ is called **least element of *A* with respect to *R*** if and only if aRx for all $x \in A$.

(iv) If R is a **partial order** on A, then the element $a \in A$ is called **greatest element of *A* with respect to *R*** if and only if xRa for all $x \in A$.

(v) If R is a **partial order** on A, then the element $a \in A$ is called **minimal element of *A* with respect to *R*** if and only if xRa then $a = x$ for all $x \in A$.

(vi) If R is a **partial order** on A, then the element $a \in A$ is called **maximal element of *A* with respect to *R*** if and only if aRx then $a = x$ for all $x \in A$.

Example 3.2.7.
(i) The relation on the set of integers $\mathbb{Z}$ defined by

$$(x, y) \in R \text{ if } x - y = 2k \text{ for some } k \in \mathbb{Z}$$

is an equivalence relation, and partitions the set integers into two equivalence classes, i.e., the even and odd integers.
If $y = 0$, then $[x] = \mathbb{Z}_e$. If $y = 1$, then $[x] = \mathbb{Z}_o$.
(ii) The inclusion relation $\subseteq$ is a partial order on the set of subsets $P(S)$ of a set S.
(iii) Let $A = \{3,6,7\}$, and

$$R_1 = \{(x, y) \in A \times A: x \leq y\}, R_2 = \{(x, y) \in A \times A: x \geq y\}$$
$$R_3 = \{(x, y) \in A \times A: y \text{ divisible by } x\}$$

are relations defined on A.

$$R_1 = \{(3,3), (3,6), (3,7), (6,6), (6,7), (7,7)\},$$
$$R_2 = \{(3,3), (6,3), (6,6), (7,3), (7,6), (7,7)\}.$$
$$R_3 = \{(3,3), (3,6), (6,6), (7,7)\}.$$

R_1,R_2 and R_3 are partial orders on A.
(1) The least element of A with respect to R_1 is .
(2) The least element of A with respect to R_2 is .
(3) The greatest element of A with respect to R_1 is .

(4) The greatest element of A with respect to R_2 is .
(5)A has no least and greatest element with respect to R_3.
(6) The maximal element of A with respect to R_3 is .
(7) The minimal element of A with respect to R_3 is .
(iv) Let $X = \{1,2,4,7\}$, $K = \{\{1,2\},\{4,7\},\{1,2,4\},X\}$ and

$$R_1 = \{(A,B) \in K \times K : A \subseteq B\},$$
$$R_2 = \{(A,B) \in K \times K : A \supseteq B\},$$

are relations defined on K.

$R_1 =$ $(\{1,2\},\{1,2\})$, $(\{1,2\},\{1,2,4\})$, $(\{1,2\},X)$,
$(\{4,7\},\{4,7\})$, $(\{4,7\},X)$,
$(\{1,2,4\},\{1,2,4\})$, $(\{1,2,4\},X)$,
(X,X)

$R_2 =$ $(\{1,2\},\{1,2\})$,
$(\{4,7\},\{4,7\})$,
$(\{1,2,4\},\{1,2\})$, $(\{1,2,4\},\{1,2,4\})$,
$(X,\{1,2\})$, $(X,\{4,7\})$, $(X,\{1,2,4\})$, (X,X)

R_1,R_2 and R_3 are partial orders on K.
(1)K has no least element with respect to R_1.
(2) The greatest element of K with respect to R_1 is .
(3) The least element of K with respect to to R_2 is .
(4)K has no greatest element with respect to R_2.
(5) The minimal elements of K with respect to R_1 are .
(6) The maximal element of K with respect to R_1 is .
(7) The minimal element of K with respect to R_2 is .
(8) The maximal element of K with respect to R_2 is .

Remark 3.2.8.
(i) Every greatest (least) element is maximal (minimal). The converse is not true.
(ii) The greatest (least) element if exist, it is unique.
(iii) every finite partially ordered set has maximal (minimal) element.
Properties of equivalence classes
(iv) $a \in [a]$.
(v) If aRb then $[a] = [b]$.
(vi) $[a] = [b] \Leftrightarrow (a, b) \in R$.
(vii) If $[a] \cap [b] \neq \emptyset$ then $[a] = [b]$.

Definition 3.2.9. R is **a totally order** on A if R is a partial order, and xRy or yRx for all $x, y \in A$, i.e. if any two elements of A are comparable with respect to R. Then we call the pair $(A, \leq)$ a **totally order set** or a **chain**.
Example 3.2.10.

(i) Let $A = \{2, 3, 4, 5, 6\}$, and define R by the usual $\leq$ relation on $\mathbb{N}$, i.e. aRb iff $a \leq b$. Then R is a **totally order** on A.

(ii) Let us define another relation on $\mathbb{N}$

$$a/b \text{ iff } a \text{ divides } b.$$

To show that / is a partial order we have to show the three defining properties of a partial order relation:
Reflexive: Since every natural number is a divisor of itself, we have a/a for all $a \in A$.
Antisymmetric: If a divides b then we have either $a = b$ or $a < b$ in the usual ordering of $\mathbb{N}$; similarly, if b divides a, then $b = a$ or $b < a$. Since $a < b$ and $b < a$ is not possible, a/b and b/a implies $a = b$.

Transitive: If a divides b and b divides c then a also divides c. Thus, / is a partial order on N.

(iii) Let $A = \{x, y\}$ and define $\leq$ on the power set $P(A)$ by

$$s \leq t \text{ iff } s \text{ is a subset of } t.$$

This gives us the following relation:

$\emptyset \leq \emptyset, \emptyset \leq \{x\}, \emptyset \leq \{y\}, \emptyset \leq \{x, y\} = A, \{x\} \leq \{x\}, \{x\} \leq \{x, y\}, \{y\} \leq \{y\}, \{y\} \leq \{x, y\}, \{x, y\} \leq \{x, y\}$.

Exercise 3.2.11.

Let $A = \{1, 2, \ldots, 10\}$ and define the relation R on A by xRy iff x is a multiple of y. Show that R is a partial order on A.