Definition 3.2.12. Inverse of a Relation
Suppose $R \subseteq A \times B$ is a relation between A and B then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by

$$bR^{-1}a \quad \text{if and only if} \quad aRb.$$

That is, $R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$.

Example 3.2.13. Let R be the relation between $\mathbb{Z}$ and $\mathbb{Z}^{+}$ defined by

$$mRn \text{ if and only if } m^2 = n.$$

Then

$$R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z}^{+}: m^2 = n\},$$

and

$$R^{-1} = \{(n,m) \in \mathbb{Z}^{+} \times \mathbb{Z} : m^2 = n\}.$$

For example, –3 R 9, –4 R 16, 16 R^{-1} 4, 9 R^{-1} 3, etc.

Remark 3.2.14.

If R is partial order relation on $A \neq \emptyset$, then R^{-1} is also partial order relation on A.

Proof.

(i) Reflexive. Let $x \in A$.

$\Longrightarrow (x,x) \in R$ (Reflexivity of A) $\Longrightarrow (x,x) \in R^{-1}$ (Def of R^{-1})

(ii) Antisymmetric. Let $(x,y) \in R^{-1}$ and $(y,x) \in R^{-1}$. To prove $x = y$.

$\Longrightarrow (y,x) \in R \wedge (x,y) \in R$ (Def of R^{-1})

$\Longrightarrow y = x$ (since R is antisymmetric).

(iii) Transitive. Let $(x,y) \in R^{-1}$ and $(y,z) \in R^{-1}$. To prove $(x,z) \in R^{-1}$.

$\Longrightarrow (y,x) \in R \wedge (z,y) \in R$ (Def of R^{-1})

$\Longrightarrow (z,x) \in R$ (since R is transitive) $\Longrightarrow (x,z) \in R^{-1}$ (Def of R^{-1}).

Definition 3.2.15. Partitions
Let A be a set and let $A_1, A_2, \ldots, A_n$ be subsets of A such
(i) $A_i \neq \emptyset$ for all i,
(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$,

(iii) $A = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$. Then the sets A_i partition the set A and these sets are called the **classes of the partition.**

Remark 3.2.16. An equivalence relation on A leads to a partition of A, and vice versa for every partition of A there is a corresponding equivalence relation.

Definition 3.2.17. The Composition of Two Relations

The composition of two relations $R_1(A, B)$ and $R_2(B, C)$ is given by $R_2 \ o \ R_1$ where

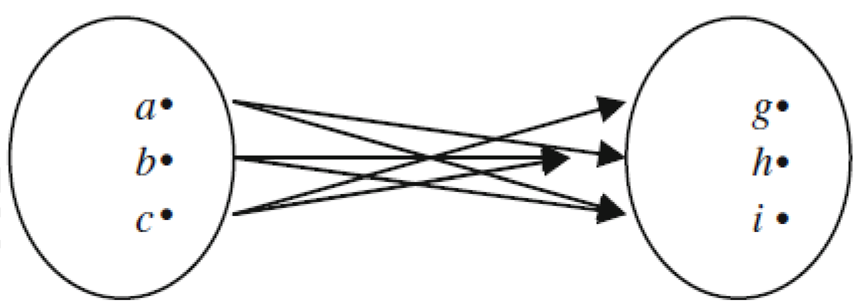
$(a, c) \in R_2 \ o \ R_1$ if and only there exists $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$.

Remark 3.2.18. The composition of relations is associative; that is,

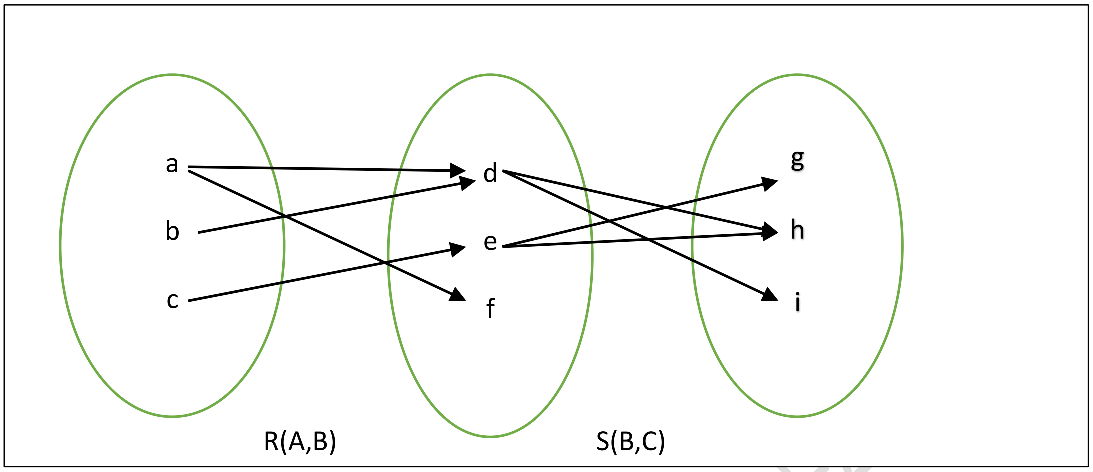
$$(\boldsymbol{R_3 \ o \ R_2}) \ \boldsymbol{o} \ \boldsymbol{R_1} = \boldsymbol{R_3 \ o} \ (\boldsymbol{R_2 \ o \ R_1})$$

Example 3.2.19.

(i) Let sets $A = \{a, b, c\}, B = \{d, e, f\}, C = \{g, h, i\}$ and relations $R(A, B) = \{(a, d), (a, f), (b, d), (c, e)\}$ and $S(B, C) = \{(d, h), (d, i), (e, g), (e, h)\}$. Then we graph these relations and show how to determine the composition pictorially S o R is determined by choosing $x \in A$ and $y \in C$ and checking if there is a route from x to y in the graph. If so, we join x to y in $S \ o \ R$.



S o R



For example, if we consider a and h we see that there is a path from a to d and from d to h and therefore (a, h) is in the composition of S and R.

(ii) Let $R^{-1} = \{(b, a)|(a, b) \in R\}$. The composition of R and R^{-1} yields:
R^{-1} o $R = \{(a, a)|a \in dom\ R\} = i_A$ and $R\ o\ R^{-1} = \{(b, b)|b \in dom\ R^{-1}\} = i_B$.

Definition 3.2.19. Union and Intersection of Relations

(i) The union of two relations $R_1(A, B)$ and $R_2\ (A, B)$ is subset of $A \times B$ and defined as

$(a, b) \in R_1 \cup R_2$ if and only if $(a, b) \in R_1$ or $(a, b) \in R_2$.

(ii) The intersection of two relations $R_1(A, B)$ and $R_2\ (A, B)$ is subset of $A \times B$ and defined as

$(a, b) \in R_1 \cap R_2$ if and only if $(a, b) \in R_1$ and $(a, b) \in R_2$.

Remark 3.2.20. The relation R_1 is a subset of R_2 $(R_1 \subseteq R_2)$ if whenever $(a, b) \in R_1$ then $(a, b) \in R_2$.