Lecture 2

The Static Equilibrium in the Atmosphere Part 2

1.1 Solved Problems

Q1. ***For the isothermal atmosphere, if the surface temperature T_o be 283 ^oK and the surface pressure p_o is 1000 hPa, calculate the pressure and the density at height of 5 km. Knowing that the density at the surface is 1.2 kg m^{-3}.***

Sol. To calculate the pressure we can use the equation:

$$p = p_o e^{-\frac{g}{RT_o}z}$$

$$p = 1000\ e^{-\frac{9.8 \times 5000}{287 \times 283}}$$

$$p = 547\ hPa$$

To calculate the temperature we may use the equation:

$$\rho = \rho_o e^{-\frac{g}{RT_o}z}$$

$$\rho = 1.2\ e^{-\frac{9.8 \times 5000}{287 \times 283}} = 0.72\ kg\ m^3$$

Q2. For homogeneous atmosphere, integrate the homogeneous equation to find an expression for the temperature as a function of height. What is the temperature at the middle of this atmosphere? (Assume the temperature at the surface is 0^o C)

Sol. From hydrostatic equation:

$$\frac{\partial p}{\partial z} = -g\rho$$

we integrate assuming that $\rho = \rho_o$:

$$\int_{p_o}^{p} dp = -g\rho_o \int_0^z dz$$

$$p - p_o = -g\ \rho_o\ z$$

$$p = p_o - g\rho_o z \qquad (1)$$

From equation of state $p = \rho_o R\ T$ we find:

$$T = \frac{p}{\rho_o R} \qquad (2)$$

Put eq1 in eq2,

$$T = \frac{p_o - \rho_o\, g\, z}{\rho_o\, R} = \frac{p_o}{\rho_o R} - \frac{\rho_o\, g\, z}{\rho_o R}$$

At the middle of the homogeneous atmosphere,

$$z = \frac{8000\, m}{2} = 4000\, m$$

$$\therefore\ T = T_o - \frac{gz}{R} \quad \Rightarrow\ T = 273 - \frac{9.8\ \times 4000}{287} = 136.4\ ^oK$$

Q3. For the isothermal atmosphere, integrate the homogeneous equation to find an expression for the pressure as a function of height.

$$T = T_o = constant$$

$$\frac{dp}{dz} = -g\rho$$

$$\rho = \frac{p}{R\, T_o}$$

$$\frac{dp}{dz} = -\frac{g\, p}{R\, T_o} \quad \Rightarrow \quad \frac{dp}{p} = -\frac{g}{R\, T_o} dz$$

$$\int_{p_o}^{p} \frac{dp}{p} = -\frac{g}{R\, T_o} \int_0^z dz$$

$$\ln \frac{p}{p_o} = -\frac{g}{RT_o} z$$

$$\frac{p}{p_o} = e^{-\frac{g}{RT_o} z}$$

$$\boxed{p = p_o\, e^{-\frac{g}{RT_o} Z}}$$

Q4. Show that the scale height for an isothermal atmosphere is equal to the height of the homogeneous atmosphere. (Assume that the temperature near the surface is 10 centigrade.

$$To\ prove\ that\ \ H_s = H$$

$$p = p_o\, e^{-\frac{g}{R\, T_o} z} \qquad (1)$$

$$p = p_o\, e^{-1} \qquad (2)$$

$$p_o e^{-\frac{g}{R\,T_o}z} = p_o\, e^{-1}$$

$$\frac{g}{R\,T_o} z = 1$$

$$\therefore H_s = \frac{R\,T_o}{g} = \frac{287\ \times 283}{9.8} = 8000\,m$$

$$\frac{dp}{dz} = -g\rho_o$$

$$\int_{p_o}^{p} dp = -\rho_o\, g\int_{0}^{z} dz \qquad (homogeneous)$$

$$p - p_o = -g\rho_o z \qquad when\ \ p = 0\,, z = H$$

$$0 = p_o - g\rho_o\, H\ \Rightarrow\ \ H = \frac{p_o}{\rho_o\, g}$$

$$H = \frac{\rho_o\, R\, T_o}{\rho_o\, g} = \frac{R\,T_o}{g} = \frac{287\times 283}{9.8} = 8000\,m$$

$$\therefore\ H_s = H = 8\,km$$

1.2 The Adiabatic Atmosphere (*3rd type of static equilibrium atmospheres* **)**

$\theta = \theta_o = constant$ (for adiabatic atmosphere)

From Poisson's equation,

$$\theta = T(\frac{p}{p_1})^{\frac{R}{C_p}}$$

and by using chain rule and differentiate both sides :

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{C_p}\,\frac{1}{p}\,\frac{\partial p}{\partial z} = 0 \qquad (How?\,Do\ it!\ \ Hint\!: use\ logarithm)$$

$$\frac{1}{T}\frac{\partial T}{\partial z} = \frac{R}{C_p}\,\frac{1}{p}\,\frac{\partial p}{\partial z}$$

$$\frac{1}{T}\frac{\partial T}{\partial z} = -\frac{g}{C_p\,T} \qquad (How?\ \ Do\ it!)$$

$$\frac{\partial T}{\partial z} = -\frac{g}{C_p} = -\frac{9.8}{1004} = 10\ {}^oK/km$$

1.3 The Hypsometric Equation

An equation relating the thickness, Δz, between two isobaric surfaces to the mean temperature of the layer. It is sometimes called the thickness equation. It is derived from the hydrostatic equation and the equation of state:

$$\frac{\partial p}{\partial z} = -g\rho \quad , \quad \rho = \frac{p}{R\,T}$$

$$\frac{\partial p}{\partial z} = -\frac{p}{R\,\overline{T}}g$$

$$\int_{z_1}^{z_2} \partial z = -\frac{R\,\overline{T}}{g}\int_{p_1}^{p_2}\frac{\partial p}{p}$$

The thickness equation is then:

$$\Delta z = \frac{R\,\overline{T}}{g}\ln\left(\frac{p_1}{p_2}\right)$$

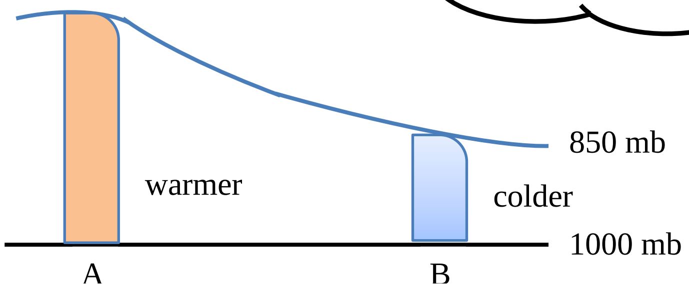

Figure (2.1) *The thickness Δz is proportional to the mean temperature of the layer*

$$\Delta z_A > \Delta z_B \quad \rightarrow \quad \overline{T}_A > \overline{T}_B$$

Q5. Show that the thickness of an isothermal atmospheric layer between 1000 mb and 800 mb can be given by:

$$\Delta z = \frac{R\,\overline{T}}{g}\ln(1.25)$$

Sol. From $\frac{\partial p}{\partial z} = -g\rho \quad , \quad \rho = \frac{p}{R\,T}$

$$\frac{\partial p}{p} = -\frac{g}{R\,\overline{T}}\partial z \quad \rightarrow \quad \int_{1000}^{800}\frac{\partial p}{p} = = -\frac{g}{R\,\overline{T}}\int_{z_1}^{z_2}\partial z$$

$$\ln 1000 - \ln 800 = \frac{g}{R\,\overline{T}}\,\Delta z$$

$$\Delta z = \frac{R\,\bar{T}}{g}\ln\frac{1000}{800} \quad = \frac{R\,\bar{T}}{g}\ln(1.25)$$

If the thickness of the (1000 – 500) mb layer is 5400 m, what is the mean temperature of the layer in centigrade?

$$\Delta z = \frac{R\,\bar{T}}{g}\ln\left(\frac{p_1}{p_2}\right)$$

$$\ln\left(\frac{p_1}{p_2}\right) = \ln\left(\frac{1000}{500}\right)$$

$$\ln 2 = 0.69$$

$$\bar{T} = \Delta z \frac{g}{R\ \ln\left(\frac{p_1}{p_2}\right)}$$

$$\bar{T} = 5400\ \times \frac{9.8}{287\ \times\ 0.69}$$

$$\bar{T} = \frac{52920}{198.93} = 266.02\ {}^{o}K$$

$$\bar{T} = 266.02 - 273 = -6.9\ {}^{o}C$$

Homework: The pilot uses the hypsometric equation to determine the height of the aircraft by taking the air temperature and air pressure data from the **control tower in the airport. Calculate the height of the aircraft knowing that the temperature is 10 Centigrade, the surface pressure is 1020 hPa, and the pressure at aircraft level measured by its device is 900 hPa.**