Chapter Four

Functions

Definition 4.1. A function or a mapping from A to B, denoted by $f: A \to B$ is a relation f from A to B in which every element from A appears exactly once as the first component of an ordered pair in the relation. That is, each $a \in A$ the relation f contains exactly one ordered pair of form (a, b) and write $f(a) = b$.

Equivalent statements to the function definition.

(i) A relation f from A to B is function iff

$$\forall x \in A \ \exists y \in B \text{ such that } (x, y) \in f$$

(ii) A relation f from A to B is function iff

$$\forall x \in A \ \forall y, z \in B, \text{ if } (x, y) \in f \land (x, z) \in f, \text{ then } y = z.$$

(iii) A relation f from A to B is function iff

$$(x_1, y_1) \text{ and } (x_2, y_2) \in f \text{ such that if } x_1 = x_2, \text{ then } y_1 = y_2.$$

This property called **the well-defined relation**.

Example 4.2.

(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$.

(1) $R_1 = \{(1, 2), (2, 4), (3, 4), (4, 5)\}$ function from A to B.

(2) $R_2 = \{(1, 2), (2, 4), (2, 5), (4, 5)\}$ not a function.

(3) $R_3 = \{(1, 2), (2, 4), (4, 5)\}$ function from $\{1, 2, 4\}$ to B.

(4) $R_4 = A \times B$ not a function.

(ii) Consider the relations described below.

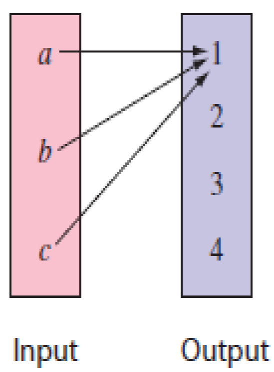
$Relation$	$Orderd\ pairs$	$Sample\ Relation$
1	$(\text{person}, \text{month})$	$\{(\text{A}, \text{May}), (\text{B}, \text{Dec}), (\text{C}, \text{Oct}), ..\}$
2	$(\text{hours}, \text{pay})$	$\{(12, 84), (4, 28), (6, 42), (15, 105), \ldots\}$
3	$(\text{instructor}, \text{course})$	$\{(\text{A}, \text{MATH001}), (\text{A}, \text{MATH002}), \ldots\}$
4	$(\text{time}, \text{temperature})$	$\{(8, 70^\circ), (10, 78^\circ), (12, 78^\circ), \ldots\}$

The first relation is a function because each person has only one birth month.
The second relation is a function because the number of hours worked at a particular job can yield only one paycheck amount.
The third relation is not a function because an instructor can teach more than one course.
The fourth relation is a function. Note that the ordered pairs $(10, 78^\circ), (12, 78^\circ)$ do not violate the definition of a function.

(iii) Decide whether each relation represents a function.

a. Input: a, b, c
Output: 2, 3, 4
$\{(a, 2), (b, 3), (c, 4)\}$

b.



c.

Input x	Output y	(x, y)
3	1	(3, 1)
4	3	(4, 3)
5	4	(5, 4)
3	2	(3, 2)

Solution.
a. This set of ordered pairs does represent a function. No first component has two different second components.
b. This diagram does represent a function. No first component has two different second components.
c. This table does not represent a function. The first component 3 is paired with two different second components, 1 and 2.

Notation 4.3. We write $f\ (a) = b$ when $(a, b) \in f$ where f is a function. We say that b is the **image** of a under f, and a is a **preimage** of b.

Definition 4.4. Let $f: A \to B$ be a function from A to B.

(i) The set A is called the **domain** of f, ($\boldsymbol{D(f)}$), and the set B is called the **codomain** of f.

(ii) The set $f\ (A)\ =\ \{f\ (x)\ |\ x\ \in\ A\}$ is called the **range** of f, ($\boldsymbol{R(f)}$).

Remark 4.5.

(i) Think of the domain as the set of possible **"input values"** for f.

(ii) Think of the range as the set of all possible **"output values"** for f.

Example 4.6.
(i) Let $A = \{p, q, r, s\}$ and $B = \{0,1,2\}$ and

$$f = \{(p, 0), (q, 1), (r, 2), (s, 2)\} \subseteq A \times B.$$

This is a function $f: A \to B$ because each element of A occurs exactly once as a first coordinate of an ordered pair in f.
We have $f\ (p) =\ 0, f\ (q) = 1, f\ (r) =\ 2$ and $f\ (s)\ = 2$. The domain of f is A, and the codomain and range are both B.

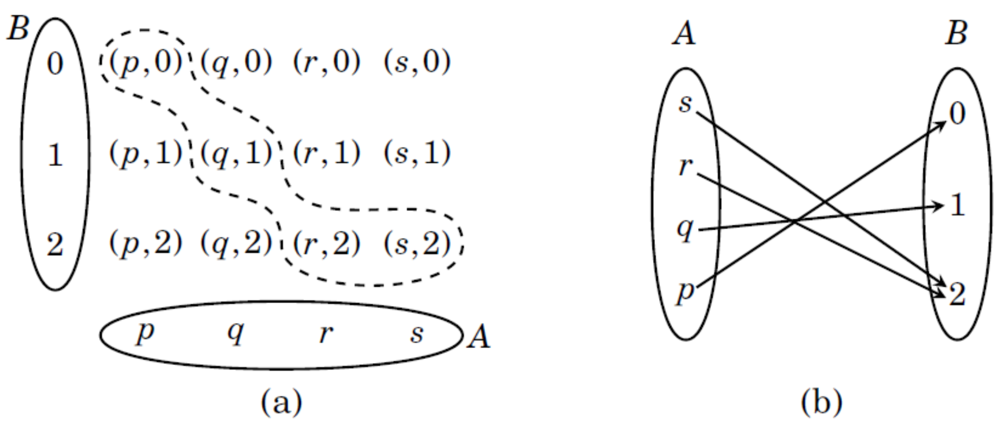


Figure. Two ways of drawing the function $f = \{(p,0),(q,1),(r,2),(s,2)\}$

(ii) Say a function $f\ :\ \mathbb{Z} \times \mathbb{Z}\ \to\ \mathbb{Z}$ is defined as $f(m, n) =\ 6m - 9n$.
Note that as a set, this function is

$$f = \{((m, n), 6m - 9n): (m, n) \in \mathbb{Z} \times \mathbb{Z}\} \subseteq (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}.$$

What is the range of ?
To answer this, first observe that for any $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, the value

$$f(m, n) =\ 6m - 9n = 3(2m - 3n)$$

is a multiple of 3. Thus every number in the range is a multiple of 3, so

$$R(f) \subseteq \{3k: k \in \mathbb{Z}\}. \qquad \cdots (1)$$

On the other hand if $b = 3k$ is a multiple of 3 we have

$$f(-k,-k) = 6(-k) - 9(-k) = -6k + 9k = 3k,$$

which means any multiple of 3 is in the range of f, so

$$\{3k: k \in \mathbb{Z}\} \subseteq R(f). \qquad \cdots (2)$$

Therefore, from (1) and (2) we get

$$R(f) = \{3k: k \in \mathbb{Z}\}.$$

Definition 4.7. Two functions $f: A \to B$ and $g: C \to D$ are **equal** if $A = C, B = D$ and $f(x) = g(x)$ for every $x \in A$.

Example 4.8.

(i) Suppose that $A = \{1,2,3\}$ and $B = \{a, b\}$. The two functions $f = \{(1, a), (2, a), (3, b)\}$ and $g = \{(3, b), (2, a), (1, a)\}$ from A to B are equal because the sets f and g are equal. Observe that the equality $f = g$ means $f(x) = g(x)$ for every $x \in A$.

(ii) Let $f(x) = (x^2 - 1)/(x - 1)$ and $g(x) = x + 1$, where $x \in \mathbb{R}$.

$f(x) = (x - 1)(x + 1)/(x - 1) = (x + 1)$.

$D(f) = \mathbb{R} - \{1\}$, $R(f) = \mathbb{R} - \{2\}$.

$D(g) = \mathbb{R}$, $R(f) = \mathbb{R}$.

$f \neq g$.