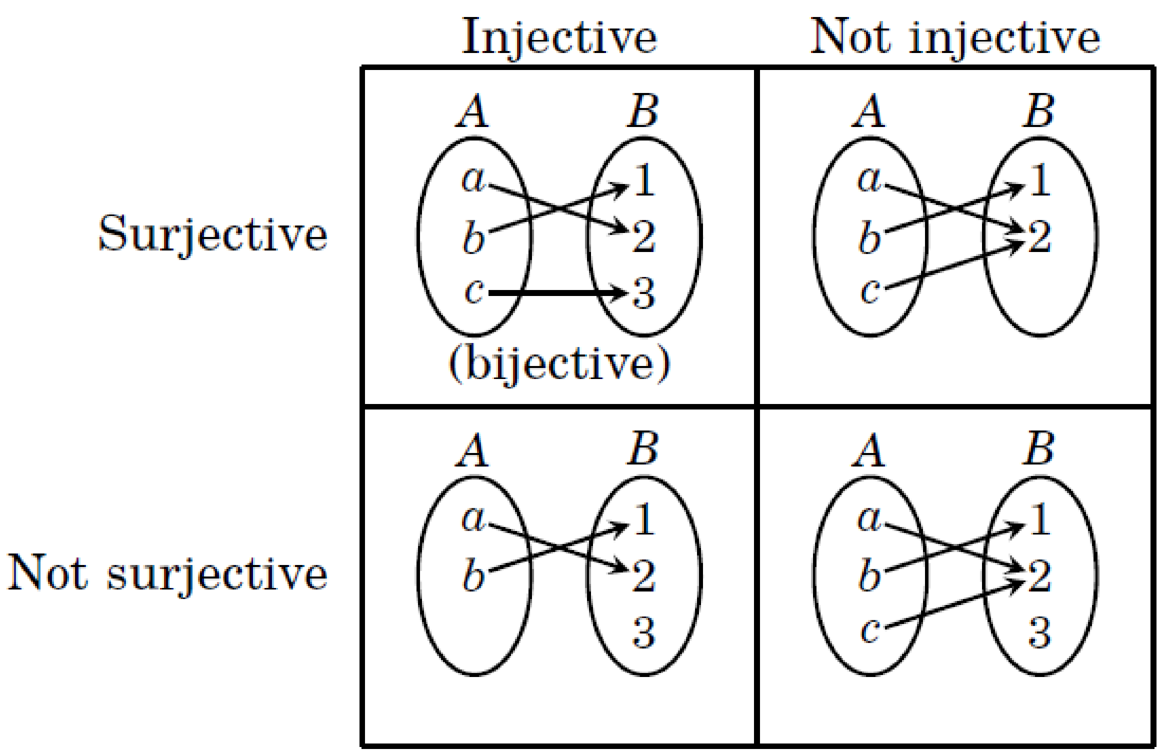
Definition 4.9.
(i) A function $f: A \to B$ is **one-to-one** or **injective** if each element of B appears at most once as the image of an element of A. That is, a function $f: A \to B$ is injective if $\forall x, y \in A,\ f(x) = f(y) \Rightarrow x = y$ **or** $\forall x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$.

(ii) A function $f: A \to B$ is **onto** or **surjective** if $f(A) = B$, that is, each element of B appears at least once as the image of an element of A. That is, a function $f: A \to B$ is surjective if $\forall y \in B\ \exists x \in A$ such that $f(x) = y$.

(iii) A function $f : A \to B$ is **bijective** iff it is one-to-one and onto.



Example 4.10. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(x) = 3x + 7$.

$$f = \{\ldots, (-3,-2), (-2,1), (-1,4), (0,7), (1,10), (2,13), \ldots\}.$$

(i) f is injective. Suppose otherwise; that is,

$$f(x) = f(y) \Rightarrow 3x + 7 = 3y + 7 \Rightarrow 3x = 3y \Rightarrow x = y$$

(ii) f is not surjective. For $b = 2$ there is no a such that $f(a) = b$; that is, $2 = 3a + 7$ holds for $a = -\frac{5}{3}$ which is not in $\mathbb{Z}$.

Example 4.11.
(i) Show that the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as
$f(x) = (1/x) + 1$ is injective but not surjective.
Solution.
We will use the contrapositive approach to show that f is injective.
Suppose $x, y \in \mathbb{R} - \{0\}$ and $f(x) = f(y)$. This means
$\frac{1}{x} + 1 = \frac{1}{y} + 1 \longrightarrow x = y$. Therefore f is injective.
Function f is not surjective because there exists an element $b = 1 \in \mathbb{R}$ for which
$f(x) = (1/x) + 1 \neq 1$ for every $x \in \mathbb{R}$.
(ii) Show that the function $f: \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by the
formula $f(m, n) = (m + n, m + 2n)$, is both injective and surjective.
Solution. Check

Definition 4.12. The **composition** of functions $f: X \to Y$ with $g: Y \to Z$ is the function $g \circ f: X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.

Remark 4.13.

(i) The composition $g \circ f$ can only be defined if the domain of g includes the range of f; that is, $R(f) \subseteq D(g)$, and the existence of $g \circ f$ does not imply that $f \circ g$ even makes sense.

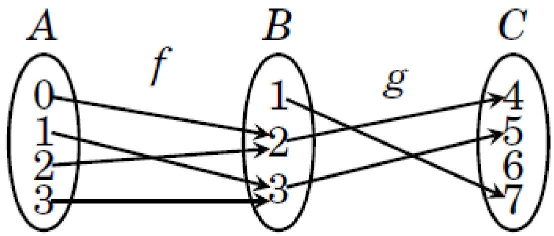
(ii) The order of application of the functions in a composition is crucial and is read from right to left.

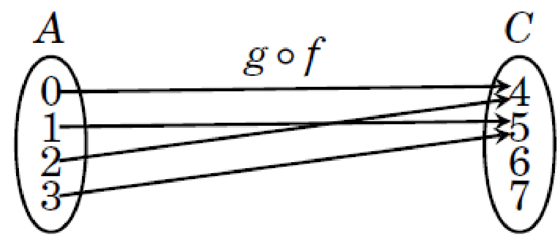
Example 4.14.

(i) Let $A = \{0,1,2,3\}$, $B = \{1,2,3\}$, $C = \{4,5,6,7\}$. If $f : A \to B$ and $g : B \to C$ are the functions defined as follows.

$f = \{(0,2), (1,3), (2,2), (3,3)\}$, $g = \{(1,7), (2,4), (3,5)\}$.

$g \circ f = \{(0,4), (1,5), (2,4), (3,5)\}$





(ii) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions defined as follows.

$f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $(g \circ f)(x) = g\left(f(x)\right) = g(x^2) = \sqrt{x^2}$.

Here $(f) = [0, \infty) \subseteq D(g) = [0, \infty)$.

Theorem 4.15.

(i) Suppose $f : A \to B$ and $g : B \to C$. If both f and g are injective, then $g \circ f$ is injective. If both f and g are surjective, then $g \circ f$ is surjective.

(ii) Composition of functions is associative. That is, if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(g \circ f) \circ h = f \circ (g \circ h)$.

Proof.

(i) To prove $g \circ f$ is 1-1. Let $x, y \in A$ and $(g \circ f)(x) = (g \circ f)(y)$. To prove $x = y$.

$(g \circ f)(x) = g\left(f(x)\right) = g(f(y))$ Def. of $\circ$

$f(x) = f(y)$ Since g is 1-1 and Def. of 1-1on g

$x = y$ Since f is 1-1 and Def. of 1-1on f

$\therefore g \circ f$ is 1-1.

To prove $g \circ f$ is onto. Let $z \in D$, to prove $\exists\, x \in A$ such that $(g \circ f)(x) = z$.

(1) $\exists y \in B$ such that $g(y) = z$ Since g is onto and Def. of onto on g

(2) $\exists x \in A$ such that $f(x) = y$ Since f is onto and Def. of onto on f

From (1) and (2) we get $g\left(f(x)\right) = z \Longrightarrow (g \circ f)(x) = z$ Def. of $\circ$

$\therefore g \circ f$ is onto.

(ii) Exercise.

Theorem 4.16. Let $f : X \to Y$ be a function. Then f is bijective iff the inverse relation f^{-1} is a function from B to A.

Proof.

Suppose $f : X \to Y$ is bijective. To prove f^{-1} is a function from B to A.
$(*)$ Let (y_1, x_1) and $(y_2, x_2) \in f^{-1}$ such that $y_1 = y_2$, to prove $x_1 = x_2$.

(x_1, y_1) and $(x_2, y_2) \in f$	Def. of f^{-1}
(x_1, y_1) and $(x_2, y_1) \in f$	By hypothesis $(*)$
$x_1 = x_2$	Def. of 1-1 on f

$\therefore f^{-1}$ is a function from B to A.

Conversely, suppose f^{-1} is a function from B to A, to prove $f : X \to Y$ is bijective, that is, 1-1 and onto.

1-1: Let $a, b \in X$ and $f(a) = f(b)$. To prove $a = b$.

$(a, f(a))$ and $(b, f(b)) \in f$	Hypothesis (f is function)
$(a, f(a))$ and $(b, f(a)) \in f$	Hypothesis ($f(a) = f(b)$)
$(f(a), a)$ and $(f(a), b) \in f^{-1}$	Def. of inverse relation f^{-1}
$a = b$	Since f^{-1} is function

$\therefore f$ is 1-1.

onto: Let $b \in Y$. To prove $\exists a \in A$ such that $f(a) = b$.

$(b, f^{-1}(b)) \in f^{-1}$	Hypothesis (f^{-1} is a function from B to A)
$(f^{-1}(b), b) \in f$	Def. of inverse relation f^{-1}

Put $a = f^{-1}(b)$.

$a \in A$ and $f(a) = b$	Hypothesis (f is function)

$\therefore f$ is onto.