Chapter One

Some Types of Functions

1.Inverse Function and Its Properties

We start this section by restate some basic and useful concepts.

Definition 1.1.1. (Inverse of a Relation)
Suppose $R \subseteq A \times B$ is a relation between A and B then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by

$$bR^{-1}a \quad \text{if and only if} \quad aRb.$$

That is, $R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$.

Definition 1.1.2. (Function)

(i) A relation f from A to B is said to be function iff

$$\forall x \in A\ \exists!\, y \in B \text{ such that } (x,y) \in f$$

(ii) A relation f from A to B is said to be function iff

$$\forall x \in A\ \forall y, z \in B, \text{ if } (x,y) \in f \wedge (x,z) \in f, \text{ then } y = z.$$

(iii) A relation f from A to B is said to be function iff

$$(x_1, y_1) \text{ and } (x_2, y_2) \in f \text{ such that if } x_1 = x_2, \text{ then } y_1 = y_2.$$

This property called **the well-defined relation**.

Notation 1.1.3. We write $f(a) = b$ when $(a,b) \in f$ where f is a function. We say that b is the **image** of a under f, and a is a **preimage** of b.

Question 1.1.4. It is clear from Definition 1.1 and 1.2 that if $f : X \to Y$ is a function, does $f^{-1}: Y \to X$ exist? If Yes, does $f^{-1}: Y \to X$ is a function?

Example 1.1.5.

(i) Let $A = \{1,2,3\}$, $B = \{a,b\}$ and f_1 be a function from A to B defined bellow. $f_1 = \{(1,a),(2,a),(3,b)\}$. Then $f_1{}^{-1}$ is ------------------------ .

(ii) Let $A = \{1,2,3\}$, $B = \{a,b,c,d\}$ and f_2 be a function from A to B defined bellow. $f_2 = \{(1,a),(2,b),(3,d)\}$. Then $f_2{}^{-1}$ is ------------------------ .

(iii) Let $A = \{1,2,3\}$, $B = \{a, b, c, d\}$ and f_3 be a function from A to B defined bellow. $f_3 = \{(1, a), (2, b), (3, a)\}$. Then ${f_3}^{-1}$is ------------------------ .

(iv) Let $A = \{1,2,3\}$, $B = \{a, b, c, \}$ and f_4 be a function from A to B defined bellow. $f_4 = \{(1, a), (2, b), (3, c)\}$. Then ${f_4}^{-1}$ is ------------------------ .

(v) Let $A = \{1,2,3\}$, $B = \{a, b, c, \}$ and f_5 be a relation from A to B defined bellow. $f_5 = \{(1, a), (1, b), (3, c)\}$. Then f_5 is -------------------- and ${f_5}^{-1}$ is ----------------- .

Definition 1.1.6. (Inverse Function)

The function $f: X \to Y$ is said to be has inverse if the inverse relation $f^{-1}: Y \to X$ is function.

Example 1.1.7.

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 3$, that is,

$$f = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x + 3\}$$

$$f = \{(x, x + 3) \in \mathbb{R} \times \mathbb{R}\}.$$

Then

$$f^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: (y, x) \in f\}$$

$$f^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x = y + 3\}$$

$$f^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x - 3\}$$

$$f^{-1} = \{(x, x - 3) \in \mathbb{R} \times \mathbb{R}\}, \text{ that is } f^{-1}(x) = x - 3.$$

f^{-1} is function as shown below.

Let $(y_1, f^{-1}(y_1))$ and $(y_2, f^{-1}(y_2)) \in f^{-1}$ such that $y_1 = y_2$, T. P. $f^{-1}(y_1) = f^{-1}(y_2)$.

Since $y_1 = y_2$, then $y_1 - 3 = y_2 - 3$ (By add -3 to both sides)

$\Longrightarrow f^{-1}(y_1) = f^{-1}(y_2)$.

(ii) $g : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, that is,

$$g = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2\}$$

$$g = \{(x, x^2) \in \mathbb{R} \times \mathbb{R}\}.$$

Then

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: (y, x) \in g\}$$

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x = y^2\}$$

$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \pm\sqrt{x}\}$

$$g^{-1} = \{(x, \pm\sqrt{x}) \in \mathbb{R} \times \mathbb{R}\}, \text{ that is } g^{-1}(x) = \pm\sqrt{x}.$$

g^{-1} is not function since $g^{-1}(4) = \pm 2$.

Theorem 1.1.8. Let $f : A \to B$ be a function. Then f is bijective iff the inverse relation f^{-1} is a function from B to A.

Proof.
Suppose $f : A \to B$ is bijective. To prove f^{-1} is a function from B to A.
$f^{-1} \neq \emptyset$ since f is onto.
$(*)$ Let (y_1, x_1) and $(y_2, x_2) \in f^{-1}$ such that $y_1 = y_2$, to prove $x_1 = x_2$.

(x_1, y_1) and $(x_2, y_2) \in f$ Def. of f^{-1}

(x_1, y_1) and $(x_2, y_1) \in f$ By hypothesis $(*)$

$x_1 = x_2$ Def. of 1-1 on f

$\therefore f^{-1}$ is a function from B to A.

Conversely, suppose f^{-1} is a function from B to A, to prove $f : A \to B$is bijective, that is, 1-1 and onto.

1-1: Let $a, b \in X$ and $f(a) = f(b)$. To prove $a = b$.

$(a, f(a))$ and $(b, f(b)) \in f$ Hypothesis ($f$is function)

$(a, f(a))$ and $(b, f(a)) \in f$ Hypothesis ($f(a) = f(b)$)

$(f(a), a)$ and $(f(a), b) \in f^{-1}$ Def. of inverse relation f^{-1}

$a = b$ Since f^{-1} is function

$\therefore f$ is 1-1.

onto: Let $b \in Y$. To prove $\exists a \in A$ such that $f(a) = b$.

$(b, f^{-1}(b)) \in f^{-1}$ Hypothesis (f^{-1} is a function from B to A)

$(f^{-1}(b), b) \in f$ Def. of inverse relation f^{-1}

Put $a = f^{-1}(b)$.

$a \in A$ and $f(a) = b$ Hypothesis (f is function)

$\therefore f$ is onto.

Definition 1.1.9. Let $f : X \to Y$ be a function and $A \subseteq X$ and $B \subseteq y$.

(i) The set $f(A) = \{f(x) \in Y: x \in A\} = \{y \in Y: \exists x \in A$ such that $y = f(x)\}$ is called the **direct image of A by f**.

(ii) The set $f^{-1}(B) = \{x \in X: f(x) \in B\}$ is called the **inverse image of B with respect to f**.

Example 1.1.10.

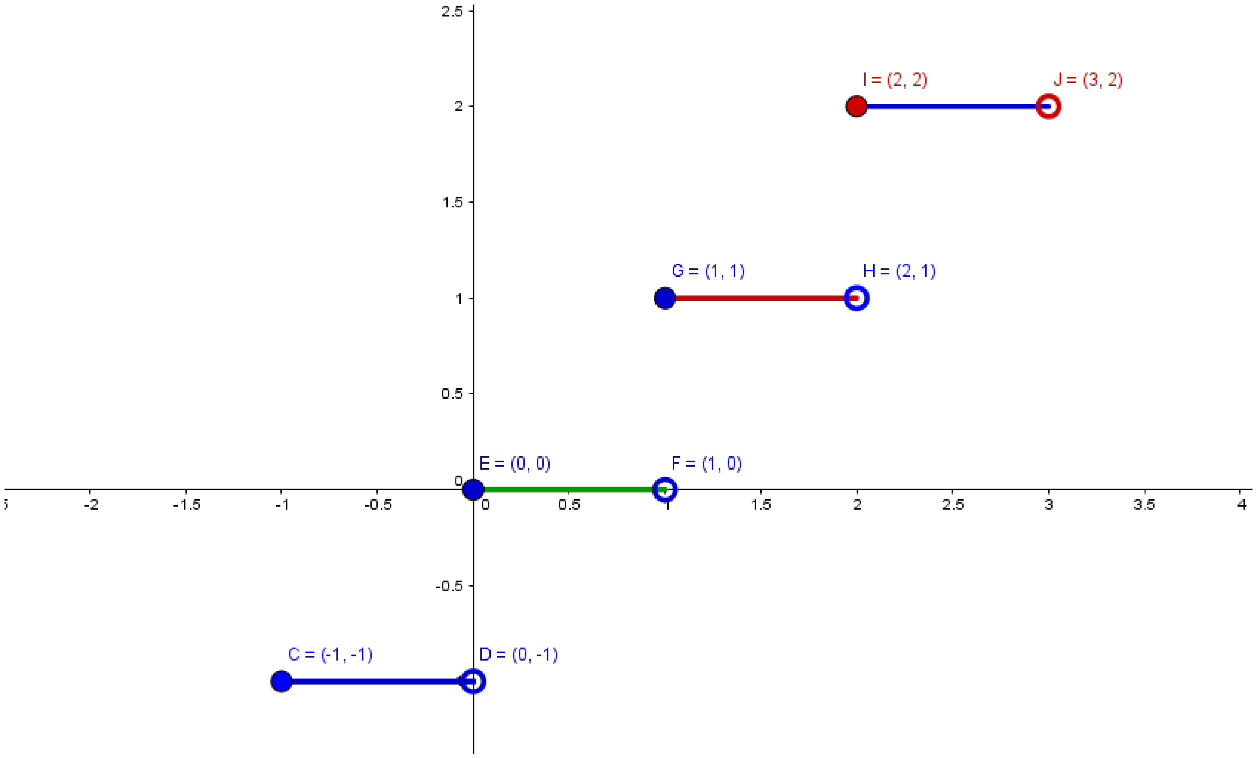
(i) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^4 - 1$. $f^{-1}(15) = \{x \in \mathbb{R}: x^4 - 1 = 15\}$

$$= \{x \in \mathbb{R}: x^4 = 16\} = \{-2,2\}.$$

(ii) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \end{cases}$. $D(f) = [-1,3), R(f) = \{-1,0,1,2\}$.

$f([-1, -1/2]) = -1$. $f([-1,0]) = \{-1,0\}$.

$f^{-1}(0) = [0,1)$. $f^{-1}([1, 3/2]) = [1,2)$.


I = (2, 2)
J = (3, 2)
G = (1, 1)
H = (2, 1)
E = (0, 0)
F = (1, 0)
C = (-1, -1)
D = (0, -1)