Definition 1.1.11.

(i) A function $I_A : A \to A$ defined by $I_A(x) = x$, for every $x \in A$ is called the **identity** function on A. $I_A = \{(x, x): x \in A\}$.

(ii) Let $A \subseteq X$. A function $i_A : A \to X$ defined by $i_A(x) = x$, for every $x \in A$ is called the **inclusion** function on A.

Theorem 1.1.12.

If $f : X \to Y$ is a bijective function, then $f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$.

Proof. Exercise.

Example 1.1.13. Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be a function defined as

$$f(m, n) = (m + n, m + 2n).$$

f is bijective(**Exercise**).

To find the inverse f^{-1} formula, let $f(n, m) = (x, y)$. Then

$(m + n, m + 2n) = (x, y)$. So, the we get the following system

$$\begin{aligned} m + n &= x \dots(1) \\ m + 2n &= y \dots(2) \end{aligned}$$

From (1) we get $m = x - n \quad \dots(3)$

$n = y - x$ Inf (2) and (3) $\dots(4)$

$m = 2x - y$ Rep (n: $y - x$) or sub(4) in (3)

Define f^{-1} as follows

$$f^{-1}(x, y) = (2x - y, y - x).$$

We can check our work by confirming that $f \circ f^{-1} = i_Y$.

$(f \circ f^{-1})(x, y) = f(2x - y, y - x)$

$$= ((2x - y) + (y - x), (2x - y) + 2(y - x))$$

$$= (x, 2x - y + 2y - 2x) = (x, y) = i_Y(x, y)$$

Remark 1.1.14. If $f : X \to Y$ is oneto-one but not onto, then one can still define an inverse function $f^{-1} : R(f) \to X$ whose domain in the range of f.

Theorem 1.1.15. Let $f : X \to Y$ be a function.

(i) If $\{Y_j \subset Y : j \in J\}$ is a collection of subsets of Y, then

$f^{-1}(\bigcup_{j\in J} Y_j) = \bigcup_{j\in J} f^{-1}(Y_j)$ and $f^{-1}\left(\bigcap_{j\in J} Y_j\right) = \bigcap_{j\in J} f^{-1}(Y_j)$

(ii) If $\{X_i \subset X: i \in I\}$ is a collection of subsets of X, then

$f(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} f(X_i)$ and $f(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} f(X_i)$.

(iii) If A and B are subsets of X such that $A = B$, then $f(A) = f(B)$. Then converse is not true.

(iv) If C and D are subsets of Y such that $C = D$, then $f^{-1}(C) = f^{-1}(D)$. Then converse is not true.

(v) If A and B are subsets of X, then $f(A) - f(B) \subseteq f(A - B)$. The converse is not true.

(vi) If C and D are subsets of Y, then $f^{-1}(C) - f^{-1}(D) = f^{-1}(C - D)$.

Proof:

(i) Let $x \in f^{-1}(\bigcup_{j\in J} Y_j)$.

$\exists y \in \bigcup_{j\in J} Y_j$ such that $f(x) = y$ Def. of inverse relation f^{-1}

$y \in Y_j$ for some $j \in J$ Def. of $\bigcup$

$x \in f^{-1}(Y_j)$ Def. of inverse f^{-1}

so $x \in \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\bigcup$

It follow that $f^{-1}(\bigcup_{j\in J} Y_j) \subseteq \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\subseteq$ $\dots\dots(*)$

Conversely, If $x \in \bigcup_{j\in J} f^{-1}(Y_j)$, then $x \in f^{-1}(Y_j)$, for some $j \in J$ Def. of $\cup$

So $f(x) \in Y_j$ and $f(x) \in \bigcup_{j\in J} Y_j$ Def. of inverse and $\cup$

$x \in f^{-1}(\bigcup_{j\in J} Y_j)$ Def. of inverse f^{-1}

It follow that $\bigcup_{j\in J} f^{-1}(Y_j) \subseteq f^{-1}(\bigcup_{j\in J} Y_j)$ Def. of $\subseteq$ $\dots\dots(**)$

$f^{-1}(\bigcup_{j\in J} Y_j) = \bigcup_{j\in J} f^{-1}(Y_j)$ From $(*), (**)$ and Def. of $=$

Example 1.1.16. Let $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function defined as $f(x) = 1$.

$\mathbb{Z}_e \cap \mathbb{Z}_o = \emptyset$. $f(\mathbb{Z}_e \cap \mathbb{Z}_o) = f(\emptyset) = \emptyset$. But $f(\mathbb{Z}_e) \cap f(\mathbb{Z}_o) = \{1\}$.

2.Types of Function

Definitions 1.2.1.

(i) (Constant Function)

The function $f: X \longrightarrow Y$ is said to be **constant function** if there exist a unique element $b \in Y$ such that $f(x) = b$ for all $x \in X$.

(ii) (Restriction Function)

Let $f: X \longrightarrow Y$ be a function and $A \subseteq X$. Then the function $g: A \longrightarrow Y$ defined by $g(x) = f(x)$ all $x \in X$ is said to be **restriction function** of f and denoted by $g = f|_A$.

(iii) (Extension Function)

Let $f: A \longrightarrow B$ be a function and $A \subseteq X$. Then the function $g: X \longrightarrow B$ defined by $g(x) = f(x)$ all $x \in A$ is said to be **extension function** of f from A to X.

(iv) (Absolute Value Function)

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ which defined as follows

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$$

is called the **absolute value function.**

(v) (Permutation Function)

Every bijection function f on a non empty set A is said to be **permutation** on A.

(vi) (Sequence)

Let A be a non empty set. A function $f: \mathbb{N} \longrightarrow A$ is called a sequence in A and denoted by $\{f_n\}$, where $f_n = f(n)$.

(vii) (Canonical Function)

Let A be a non empty set, R an equivalence relation on A and A/R be the set of all equivalence class. The function $\pi: A \longrightarrow A/R$ defined by $\pi(x) = [x]$ is called the **canonical function**.

(viii) (Projection Function)

Let A_1, A_2 be two sets. The function $P_1: A_1 \times A_2 \longrightarrow A_1$ defined by $P_1(x, y) = x$ for all $(x, y) \in A_1 \times A_2$ is called the **first projection.**

The function $P_2: A_1 \times A_2 \longrightarrow A_2$ defined by $P_2(x, y) = y$ for all $(x, y) \in A_1 \times A_2$ is called the **second projection.**

(ix) (Cross Product of Functions)

Let $f: A_1 \longrightarrow A_2$ and $g: B_1 \longrightarrow B_2$ be two functions. The cross product of f with g, $f \times g: A_1 \times B_1 \longrightarrow A_2 \times B_2$ is the function defined as follows:

$$(f \times g)(x, y) = (f(x), g(y)) \text{ for all } (x, y) \in A_1 \times B_1.$$