Foundation of Mathematics II

Chapter Two System of Numbers

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2017-2018

Chapter Two

System of Numbers

1. Natural Numbers

Let $0 =$Set with no point, that is; $0 = \emptyset$, $1 =$Set with one point, that is; $1 = \{0\}$, $2 =$Set with two points, that is; $2 = \{0,1\}$, and so on. Therefore,

$1 = \{0\} = \{\emptyset\}$,

$2 = \{0,1\} = \{\emptyset, \{\emptyset\}\}$,

$3 = \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$,

$4 = \{0,1,2,3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$,

$\vdots$

$n = \{0,1,2,3, \dots, n-1\}$.

Definition 2.1.1. Let A be a set. A **successor** to A is $A^+ = A \cup \{A\}$ and denoted by A^+.

According to above definition we can get the numbers $0,1,2,3,\dots$ as follows:

$0 = \emptyset$,

$1 = \{0\} = \emptyset \cup \{\emptyset\} = \emptyset^+ = 0^+$,

$2 = \{0,1\} = \{0\} \cup \{1\} = 1 \cup \{1\} = 1^+$,

$3 = \{0,1,2\} = \{0,1\} \cup \{2\} = 2 \cup \{2\} = 2^+$,

Definition 2.1.2. A set A is said to be **successor set** if it satisfies the following conditions:

(i) $\emptyset \in A$,

(ii) if $a \in A$, then $a^+ \in A$.

Remark 2.1.3.

(i) Any successor set should contains the numbers $0,1,2,\dots n$.

(ii) Collection of all successor sets is not empty.

(iii) Intersection of any non empty collection of successor sets is also successor set.

Definition 2.1.4. Intersection of all successor sets is called **the set of natural numbers** and denoted by $\mathbb{N}$, and each element of $\mathbb{N}$ is called **natural element**.

Peano's Postulate 2.1.5.

(P$_1$) $0 \in \mathbb{N}$.
(P$_2$) If $a \in \mathbb{N}$, then $a^+ \in \mathbb{N}$.
(P$_3$) $0 \neq a^+ \in \mathbb{N}$ for every natural number a.
(P$_4$) If $a^+ = b^+$, then $a = b$ for any natural numbers a, b.
(P$_5$) If X is a successor subset of $\mathbb{N}$, then $X = \mathbb{N}$.

Remark 2.1.6.
(i) **P$_1$** says that 0 should be a natural number.
(ii) **P$_2$** states that the relation $+: \mathbb{N} \longrightarrow \mathbb{N}$, defined by $+(n) = n^+$ is mapping.
(iii) **P$_3$** as saying that 0 is the first natural number, or that '-1' is not an element of $\mathbb{N}$.
(iv) **P$_4$** states that the map $+: \mathbb{N} \longrightarrow \mathbb{N}$ is injective.
(v) **P$_5$** is called the **Principle of Induction**.

2.1.7. Addition + on $\mathbb{N}$

We will now define the operation of addition $+$ using only the information provided in the Peano's Postulates.

Let $a, b \in \mathbb{N}$. We define $+: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ as follows:

$$+(\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{a}+\boldsymbol{b} = \begin{cases} \boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a} & \boldsymbol{if\ b = 0} \\ \boldsymbol{a} + \boldsymbol{c}^+ = (\boldsymbol{a} + \boldsymbol{c})^+ & \boldsymbol{if\ b \neq 0} \end{cases}$$

where $\boldsymbol{b} = \boldsymbol{c}^+$.

Therefore, if we want to compute $1 + 1$, we note that $1 = 0^+$ and get

$$1 + 1 = 1 + 0^+ = (1 + 0)^+ = 1^+ = 2.$$

We can proceed further to compute $1 + 2$.
To do so, we note that $2 = 1^+$ and therefore that

$$1 + 2 = 1 + 1^+ = (1 + 1)^+ = 2^+ = 3.$$

2.1.8. Multiplication $\cdot$ on $\mathbb{N}$

We will now define the operation of multiplication $\cdot$ using only the information provided in the Peano's Postulates.

Let $a, b \in \mathbb{N}$. We define $+: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ as follows:

$$\cdot(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} \boldsymbol{a} \cdot \mathbf{0} = \mathbf{0} & \boldsymbol{if\ b = 0} \\ \boldsymbol{a} \cdot \boldsymbol{c}^+ = \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} & \boldsymbol{if\ b \neq 0} \end{cases}$$

where $\boldsymbol{b} = \boldsymbol{c}^+$.

Thus, we can easily show that $a \cdot 1 = a$ by noting that $1 = 0^+$ and therefore

$$a \cdot 1 = a \cdot 0^+ = a + (a \cdot 0) = a + 0 = a.$$

We can use this to multiply $3 \cdot 2$. Of course, we know that $2 = 1^+$ and therefore

$$3 \cdot 2 = 3 \cdot 1^+ = 3 + (3 \cdot 1) = 3 + 3 = 6.$$

Remark 2.1.9. From 2.1.7 and 2.1.8 we can deduce that for all $n \in \mathbb{N}$, if $n \neq 0$, then there exist an element $m \in \mathbb{N}$ such that $n = m^+$.

Theorem 2.1.10.

(i) $n^+ = n + 1, \forall\, n \in \mathbb{N}$.

(ii) $n + m = m + n, \forall\, n, m \in \mathbb{N}$. (Commutative property of $+$)

(iii) $(n + m) + c = n + (m + c), \forall\, n, m, c \in \mathbb{N}$. (Associative property of $+$)

(iv) $n \cdot m = m \cdot n, \forall\, n, m \in \mathbb{N}$. (Commutative property of $\cdot$)

(v) $(n \cdot m) \cdot c = n \cdot (m \cdot c), \forall\, n, m, c \in \mathbb{N}$. (Associative property of $\cdot$)

(vi) $(n + m) \cdot c = n \cdot c + m \cdot c, \forall\, n, m, c \in \mathbb{N}$. (Distributive property of $\cdot$ on $+$)
$c \cdot (n + m) = c \cdot n + c \cdot m$.

(vii) The addition operation $+$ defined on $\mathbb{N}$ is unique.

(viii) The addition operation $\cdot$ defined on $\mathbb{N}$ is unique.

(ix) (Cancellation Law for $+$). $m + c = n + c$, for some $c \in \mathbb{N} \Leftrightarrow m = n$.

(x) (Cancellation Law for $\cdot$). $m \cdot c = n \cdot c$, for some $c (\neq 0) \in \mathbb{N} \Leftrightarrow m = n$.

(xi) 0 is the unique element such that $0 + m = m + 0 = m, \forall\, m \in \mathbb{N}$.

(xii) 1 is the unique element such that $1 \cdot m = m \cdot 1 = m, \forall\, m \in \mathbb{N}$.

(xiii) If $m \cdot n = 0$, then either $m = 0$ or $n = 0, \forall\, m, n \in \mathbb{N}$. ($\mathbb{N}$ has no zero divisor)

Proof:

(i) $n^+ = (n + 0)^+$ (Since $n = n + 0$)
$= n + 0^+$ (Def. of $+$)
$= n + 1$ (Since $0^+ = 1$)

(ii) Suppose that $L_m = \{n \in \mathbb{N} | m + n = n + m\}, m \in \mathbb{N}$. Then prove that L_m is successor subset of $\mathbb{N}$.

(iii) Let $L_{mn} = \{c \in \mathbb{N} | (m+n)+c = m+(n+c)\}, m, n \in \mathbb{N}$.
(1) $(m+n)+0 = m+n = m+(n+0)$; that is, $0 \in L_{mn}$. Therefore, $L_{mn} \neq \emptyset$.
(2) Let $c \in L_{mn}$; that is, $(m+n)+c = m+(n+c)$. To prove $c^+ \in L_{mn}$.

$$\begin{aligned}(m+n)+c^+ &= ((m+n)+c)^+ \\ &= (m+(n+c))^+ \quad (\text{since } c \in L_{mn}) \\ &= m+(n+c)^+ \quad (\text{Def. of } +) \\ &= m+(n+c^+) \quad (\text{Def. of } +)\end{aligned}$$

Thus, $c^+ \in L_{mn}$. Therefore, L_{mn} is a successor subset of $\mathbb{N}$. So, we get by **P**$_5$ $L_{mn} = \mathbb{N}$.

(iv) Suppose that $L_m = \{n \in \mathbb{N} | m \cdot n = n \cdot m\}, m \in \mathbb{N}$. Then prove that L_m is successor subset of $\mathbb{N}$.

(v) Suppose that $L_{mn} = \{c \in \mathbb{N} | (m+n) \cdot c = m \cdot c + n \cdot c\}, m, n \in \mathbb{N}$. Then prove that L_{mn} is successor subset of $\mathbb{N}$.

(vi) Suppose that $L_{mn} = \{c \in \mathbb{N} | c \cdot (m+n) = c \cdot m + c \cdot n\}, m, n \in \mathbb{N}$. Then prove that L_{mn} is successor subset of $\mathbb{N}$.

(vii) Let $\oplus$ be another operation on such that

$$\oplus(\boldsymbol{a}, \boldsymbol{b}) = \begin{cases} \boldsymbol{a} \oplus \boldsymbol{0} = \boldsymbol{a} & \boldsymbol{if\ b = 0} \\ \boldsymbol{a} \oplus \boldsymbol{c}^+ = (\boldsymbol{a} \oplus \boldsymbol{c})^+ & \boldsymbol{if\ b \neq 0} \end{cases}$$

where $\boldsymbol{b} = \boldsymbol{c}^+$**.**

Let $L = \{m \in \mathbb{N} | n+m = n \oplus m, \forall n \in \mathbb{N}\}$.
(1) To prove $0 \in L$**.**
$n+0 = n = n \oplus 0$. Thus, $0 \in L$.
(2) To prove that $k^+ \in L$ for every $k \in L$. Suppose $k \in L$.

$$\begin{aligned} n+k^+ &= (n+k)^+ && \text{Def. of } + \\ &= (n \oplus k)^+ && (\text{Since } k \in L) \\ &= n \oplus k^+ && \text{Def. of } \oplus \end{aligned}$$

Thus, $k^+ \in L$.

From (1), (2) we get that L is a successor set and $L \subseteq \mathbb{N}$. From **P**$_5$ we get that $L = \mathbb{N}$.
(viii) Exercise.
(ix) Suppose that

$L = \{c \in \mathbb{N} | m + c = n + c,$ for some $c \in \mathbb{N} \Leftrightarrow m = n\}$, $m, n \in \mathbb{N}$. Then prove that L is successor subset of $\mathbb{N}$.

(x) Suppose that
$L = \{c \in \mathbb{N} | m \cdot c = n \cdot c,$ for some $c(\neq 0) \in \mathbb{N} \Leftrightarrow m = n\}$, $m, n \in \mathbb{N}$. Then prove that L is successor subset of $\mathbb{N}$.

(xiii) we will prove the equivalent statement that: if $n \neq 0$ and $m \neq 0$, then $m \cdot n \neq 0$.

Assume that $m \cdot n = 0$

$\longrightarrow m \cdot n = m \cdot 0$ (Def. of $\cdot$)

$\longrightarrow m = 0$ (Cancellation law of $\cdot$)

$\longrightarrow$ Contradiction since $m \neq 0$.

$\longrightarrow \therefore m \cdot n \neq 0$.

(xi),(xii)Exercise.

Definition 2.1.11. Let $x, y \in \mathbb{N}$. We say that ***x* less than *y*** and denoted by $x < y$ iff there exist $k \neq 0 \in \mathbb{N}$ such that $x + k = y$.

Theorem 2.1.12.

(i) The relation $<$ is transitive relation on $\mathbb{N}$.

(ii) $0 < n^+$ and $n < n^+$ for all $n \in \mathbb{N}$.

(iii) $0 < m$ or $m = 0$, for all $m \in \mathbb{N}$.

Proof.

(i),(ii),(iii) Exercise.

Theorem 2.1.13.(Trichotomy)

For each $m, n \in \mathbb{N}$ one and only one of the following is true:
(1) $m < n$ or (2) $n < m$ or (3) $m = n$.

Proof.

Let $m \in \mathbb{N}$ and
$L_1 = \{n \in \mathbb{N} | n < m\}$,
$L_2 = \{n \in \mathbb{N} | m < n\}$,
$L_3 = \{n \in \mathbb{N} | n = m\}$,
$M = L_1 \cup L_2 \cup L_3$.

(1) $L_i \neq \emptyset$ and $L_i \subseteq \mathbb{N}$, $i = 1,2,3$. Therefore, $M \subseteq \mathbb{N}$ and $M \neq \emptyset$.

(2) To prove that M is a successor set.

(i) To prove that $0 \in M$.

(a) If $m = 0$, then $0 \in L_3 \longrightarrow 0 \in M$ (Def. of $\cup$)

(b) If $m \neq 0$, then $\exists\ k \in \mathbb{N} \ni$
$m = k^+$

If $k \neq 0 \longrightarrow \ m = k + 1 = 1 + k$ (Commutative of $+$)
$\longrightarrow 1 < m$ (Def. of $<$)
$\longrightarrow 0 < 0^{+} = 1 < m$ (Theorem 2.1.11(ii))
$\longrightarrow 0 < m$ (Since $<$ is transitive)
$\longrightarrow 0 \in L_1$ (Def. of L_1)
$\longrightarrow 0 \in M$ (Def. of $\cup$)
If $k = 0 \longrightarrow \ m = k + 1 = 0 + 1 = 1 = 0^{+}$
$\longrightarrow 0 < 0^{+} = 1 = m \longrightarrow 0 < m$ (Theorem 2.1.11(ii))
$\longrightarrow 0 \in L_1$ (Def. of L_1)
$\longrightarrow 0 \in M$ (Def. of $\cup$)

(ii) Suppose that $k \in M$. To prove that $k^{+} \in M$.
Since $k \in M$, then $k \in L_1$ or $k \in L_2$ $k \in L_3$ (Def. of $\cup$)
(a) If $k \in L_1$
$\longrightarrow k < m$ (Def. of L_1)
$\longrightarrow \exists c \neq 0 \in \mathbb{N} \ni m = k + c$ (Def of $<$)
$\longrightarrow \exists l \neq 0 \in \mathbb{N} \ni c = l^{+}$ (Remark 2.1.9)
$\longrightarrow m = k + c = k + l^{+} = (k + l)^{+}$ (Def. of $+$)
$\longrightarrow m = (k + l)^{+} = (l + k)^{+}$ (Commutative law for $+$)
$\longrightarrow m = l + k^{+}$ (Def. of $+$)
$\longrightarrow \ k^{+} < m$ (Def. of $<$)
$\longrightarrow \ k^{+} \in L_1$ (Def. of L_1)
$\longrightarrow \ k^{+} \in M$ (Def. of $\cup$)

(b) If $k \in L_2$
$\longrightarrow m < k$ (Def. of L_2)
$\longrightarrow m < k < k^{+}$ (Theorem 2.1.12(ii))
$\longrightarrow m < k^{+}$ (Theorem 2.1.12(i))
$\longrightarrow \ k^{+} \in L_2$ (Def. of L_2)
$\longrightarrow \ k^{+} \in M$ (Def. of $\cup$)

(c) If $k \in L_3$
$\longrightarrow m = k$ (Def. of L_2)
$\longrightarrow m = k < k^{+}$ (Theorem 2.1.12(ii))
$\longrightarrow m < k^{+}$ (Theorem 2.1.12(i))
$\longrightarrow \ k^{+} \in L_2$ (Def. of L_2)
$\longrightarrow \ k^{+} \in M$ (Def. of $\cup$)

Theorem 2.1.14.

(i) For all $n \in \mathbb{N}$, $0 < n \Leftrightarrow n \neq 0$.

(ii) For all $m, n \in \mathbb{N}$, if $n \neq 0$, then $m + n \neq 0$.

(iii) $m + k < n + k \Leftrightarrow m < n$, for all $m, n, k \in \mathbb{N}$.

(iv) For all $k(\neq 0) \in \mathbb{N}$, if $m < n$, then $m \cdot k < n \cdot k$, for all $m, n \in \mathbb{N}$.

(v) For all $k(\neq 0) \in \mathbb{N}$, if $m \cdot k < n \cdot k$, then $m < n$, for all $m, n \in \mathbb{N}$.

Proof.

(ii) Case 1:

If $m = 0$.

$\longrightarrow m + n = 0 + n = n \neq 0$

$\longrightarrow m + n \neq 0$

Case 2:

If $m \neq 0 \longrightarrow 0 < m$ By (i)

Suppose that $m + n = 0$

$\longrightarrow m < 0$

$\longrightarrow m < 0$ and $0 < m$

Contradiction with Trichotomy Theorem; that is , $m + n \neq 0$.

(v) Let $m \cdot k < n \cdot k$. Assume that $m \nless n$

$\longrightarrow n < m$ or $n = m$	(Trichotomy Theorem)
Suppose $n = m$	
$\longrightarrow m \cdot k = n \cdot k$	(Cancelation law of $\cdot$)
$\longrightarrow m \cdot k = n \cdot k$ and $m \cdot k < n \cdot k$	
$\longrightarrow$ Contradiction with (Trichotomy Theorem)	
Suppose $n < m$	
$\longrightarrow n \cdot k < m \cdot k$	(From (iv)
$\longrightarrow n \cdot k < m \cdot k$ and $m \cdot k < n \cdot k$	
$\longrightarrow$ Contradiction with Trichotomy Theorem	
$\longrightarrow \therefore m < n$	

(i),(iii),(iv) Exercise.

2. Construction of Integer Numbers

Let write $\mathbb{N} \times \mathbb{N}$ as follows:

$$\mathbb{N} \times \mathbb{N} = \left\{ \begin{array}{ccccccccc} (0,0) & (0,1) & (0,2) & (0,3) & (0,4) & \cdots & \cdots & \cdots & \cdots \\ (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & \cdots & \cdots & \cdots & \cdots \\ (2,0) & (2,1) & (2,2) & (2,3) & (2,4) & \cdots & \cdots & \cdots & \cdots \\ (3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots & \cdots & \cdots & \cdots \\ (4,0) & (4,1) & (4,2) & (4,3) & (4,4) & \cdots & \cdots & \cdots & \cdots \\ (5,0) & (5,1) & (5,2) & (5,3) & (5,4) & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \end{array} \right\}$$

Let define a relation on $\mathbb{N} \times \mathbb{N}$ as follows:

$$\boxed{(a,b)R^*(c,d) \Leftrightarrow a + d = b + c}.$$

Example 2.2.1. $(1,0)R^*(4,3)$ since $1 + 3 = 0 + 4$.
$(1,0)\not{R}^*(6,4)$ since $1 + 4 \neq 0 + 6$.

Theorem 2.2.2. The relation R^* on $\mathbb{N} \times \mathbb{N}$ is an equivalence relation.

Proof.

(1) Reflexive. For all $(a,b) \in \mathbb{N} \times \mathbb{N}$, $a + b = a + b$; that is $(a,b)R^*(a,b)$.

(2) Symmetric. Let $(a,b),(c,d) \in \mathbb{N} \times \mathbb{N}$ such that $(a,b)R^*(c,d)$. To prove that $(c,d)R^*(a,b)$.

$\longrightarrow a + d = b + c$ (Def. of R^*)
$\longrightarrow d + a = c + b$ (Comm. law for $+$)
$\longrightarrow c + b = d + a$ (Equal properties)
$\longrightarrow (c,d)R^*(a,b)$ (Def. of R^*)

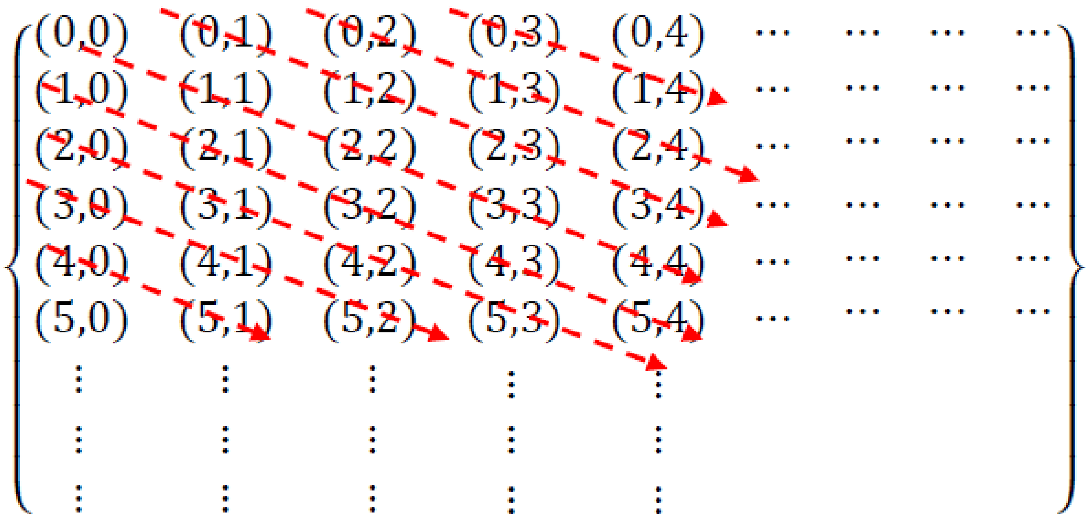
(3) Transitive. Let $(a,b),(c,d),(r,s) \in \mathbb{N} \times \mathbb{N}$ such that $(a,b)R^*(c,d)$ and $(c,d)R^*(r,s)$. To prove $(a,b)R^*(r,s)$.

$a + d = b + c$ (Since $(a,b)R^*(a,b)$) …..(1)
$c + s = d + r$ (Since $(c,d)R^*(r,s)$) …..(2)
$\longrightarrow (a + d) + s = (b + c) + s$ (Add s to both side of (1))
$= b + (c + s)$ (Cancellations low and asso. law for $+$) …..(3)
$\longrightarrow (a + d) + s = b + (c + s)$ (Sub.(2) in (3))
$= b + (d + r)$

$\longrightarrow a + (d + s) = b + (r + d)$ (Asso. law and comm. law for $+$)
$\longrightarrow a + (s + d) = b + (r + d)$ (Comm. law for $+$)
$\longrightarrow (a + s) + d = (b + r) + d$ (Asso. law for $+$)
$\longrightarrow (a + s) = (b + r)$ (Cancellation low for $+$)
$\longrightarrow (a, b)R^*(r, s)$ (Def. of R^*)

Remark 2.2.2.
(i) The equivalence class of each $(a, b) \in \mathbb{N} \times \mathbb{N}$ is as follows:

$$\boxed{[(a, b)] = [a, b] = \{(r, s) \in \mathbb{N} \times \mathbb{N} | a + s = b + r\}}.$$



$[1,0] = \{(x, y) \in \mathbb{N} \times \mathbb{N} | 1 + y = 0 + x\}$
$\quad = \{(x, y) \in \mathbb{N} \times \mathbb{N} | x = 1 + y\}$
$\quad = \{(y + 1, y) | y \in \mathbb{N}\}$
$\quad = \{(1,0), (2,1), (3,2), \ldots\}.$
$[0,0] = \{(x, y) \in \mathbb{N} \times \mathbb{N} | 0 + y = 0 + x\}$
$\quad = \{(x, y) \in \mathbb{N} \times \mathbb{N} | x = 1\}$
$\quad = \{(x, x) | x \in \mathbb{N}\}$
$\quad = \{(0,0), (1,1), (2,2), \ldots\}.$
(ii) $[a, b] = \{(a, b), (a + 1, b + 1), (a + 2, b + 2), \ldots\}.$

(iii) These classes $[(a, b)]$ formed a partition on $\mathbb{N} \times \mathbb{N}$.

Theorem 2.2.3. For all $(x, y) \in \mathbb{N} \times \mathbb{N}$, one of the following hold:
(i)$[x, y] = [0,0]$,
(ii)$[x, y] = [z, 0]$, for some $z \in \mathbb{N}$,
(iii)$[x, y] = [0, z]$, for some $z \in \mathbb{N}$.

Proof.

Let $(x,y) \in \mathbb{N} \times \mathbb{N}$. Then by Trichotomy Theorem, there are three possibilities.

(1) $x = y$,

$\longrightarrow 0 + y = 0 + x$ Def. of $+$

$\longrightarrow (0,0)R^{*}(x,y)$ Def. of R^{*}

$\longrightarrow [0,0] = [x,y]$ Def. of $[a,b]$

(2) $x < y$,

$\longrightarrow y = x + z$ for some $z \in \mathbb{N}$ Def. of $<$

$\longrightarrow x + z = y + 0$ Def. of $+$

$\longrightarrow (x,y)R^{*}(0,z) \longrightarrow (0,z)R^{*}(x,y)$ Def. of R^{*}

$\longrightarrow [0,z] = [x,y]$ Def. of $[a,b]$

(3) $y < x$,

$\longrightarrow x = y + z$ for some $z \in \mathbb{N}$ Def. of $<$

$\longrightarrow x + 0 = y + z$ Def. of $+$

$\longrightarrow (x,y)R^{*}(z,0) \longrightarrow (z,0)R^{*}(x,y)$ Def. of R^{*}

$\longrightarrow [z,0] = [x,y]$ Def. of $[a,b]$

2.2.3. Constriction of Integer Numbers $\mathbb{Z}$.

Let

$$\mathbb{Z} = \bigcup_{(a,b)\in\mathbb{N}\times\mathbb{N}} [(a,b)] = \bigcup_{a(\neq 0)\in\mathbb{N}} [(a,0)] \bigcup_{b(\neq 0)\in\mathbb{N}} [(0,b)] \bigcup [(0,0)].$$

2.2.4.Addition, Subtraction and Multiplication on $\mathbb{Z}$

Addition: $\oplus : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$**;**

$$\boxed{[r,s] \oplus [t,u] = [r + t, s + u]}$$

Subtraction: $\ominus : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$**;**

$$\boxed{[r,s] \ominus [t,u] = [r,s] \oplus [u,t] = [r + u, s + t]}$$

Multiplication: $\odot : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$**;**

$$\boxed{[r,s] \odot [t,u] = [r \cdot t + s \cdot u, r \cdot u + s \cdot t]}$$

Theorem 2.2.5. The relations $\oplus$, $\ominus$ and $\odot$ are well defined; that is, $\oplus$ and $\ominus$ is function.

Proof.

To prove $\oplus$ is function. Assume that $[r,s] = [r_0,s_0]$ and $[t,u] = [t_0,u_0]$.
$[r,s] \oplus [t,u] = [r + t, s + u]$
$[r_0,s_0] \oplus [t_0,u_0] = [r_0 + t_0, s_0 + u_0]$
To prove $[r + t, s + u] = [r_0 + t_0, s_0 + u_0]$.

$\longrightarrow (r,s)R^*(r_0,s_0)$ $[r,s] = [r_0,s_0]$ and Def. of R^*
$\longrightarrow r + s_0 = s + r_0$ ……(1)
$\longrightarrow (t,u)R^*(t_0,u_0)$ $[r,s] = [r_0,s_0]$ and Def. of R^*
$\longrightarrow t + u_0 = u + t_0$ ……(2)
$\longrightarrow (r + s_0) + (t + u_0) = (s + r_0) + (u + t_0)$ Adding (1) ,(2)
$\longrightarrow (r + t) + (s_0 + u_0) = (s + u) + (r_0 + t_0)$ Asso. and comm. for +
$\longrightarrow (r + t, s + u)R^*(r_0 + t_0, s_0 + u_0)$ Def. of R^*
$\longrightarrow [r + t, s + u] = [r_0 + t_0, s_0 + u_0]$ Def. of $[a,b]$

$\ominus$ and $\odot$**(Exercise)**

Example 2.2.7.
$[2,4] \oplus [0,1] = [2 + 0, 4 + 1] = [2,4] = [0,2]$.
$[5,2] \oplus [8,1] = [5 + 8, 2 + 1] = [13,3] = [10,0]$.

Notation 2.2.7.
(i) Let identify the equivalence classes $[r,s]$ according to its form as in Theorem 2.2.3.
$[a,0] = +a$, $a \in \mathbb{N}$, called **positive integer**.
$[0,b] = -b$, $b \in \mathbb{N}$, called **negative integer**.
$[0,0] = 0$, called the **zero element**.

$[4,6] = [0,2] = -2$
$[9,6] = [3,0] = 3$
$[6,6] = [0,0] = 0$

(ii) The relation $i: \mathbb{N} \longrightarrow \mathbb{Z}$, defined by $i(n) = [n,0]$ is 1-1 function, and $i(n + m) = i(n) \oplus i(m)$, $i(n \cdot m) = i(n) \odot i(m)$. So, we can identify n with $+n$; that is, $\boxed{+n = n}$, $\boxed{+ = \oplus}$ and $\boxed{\cdot = \odot}$.

Theorem 2.2.8.
(i) $a \in \mathbb{Z}$ is positive if there exist $[x,y] \in \mathbb{Z}$ such that $a = [x,y]$ and $y < x$.
(ii) $b \in \mathbb{Z}$ is negative if there exist $[x,y] \in \mathbb{Z}$ such that $b = [x,y]$ and $x < y$.
(iii) $(-m) \odot n = -(m \cdot n)$,
(iv) $m \odot (-n) = -(m \cdot n)$,
(v) $(-m) \odot (-n) = m \cdot n$.
(vi) $n + m = m + n, \forall\, n, m \in \mathbb{Z}$. (Commutative property of $+$)
(vii) $(n + m) + c = n + (m + c), \forall\, n, m, c \in \mathbb{Z}$. (Associative property of $+$)
(viii) $n \cdot m = m \cdot n, \forall\, n, m \in \mathbb{Z}$. (Commutative property of $\cdot$)
(ix) $(n \cdot m) \cdot c = n \cdot (m \cdot c), \forall\, n, m, c \in \mathbb{Z}$. (Associative property of $\cdot$)
(x) (Cancellation Law for $+$). $m + c = n + c$, for some $c \in \mathbb{N} \Leftrightarrow m = n$.
(xi) (Cancellation Law for $\cdot$). $m \cdot c = n \cdot c$, for some $c(\neq 0) \in \mathbb{N} \Leftrightarrow m = n$.
(xii) 0 is the unique element such that $0 + m = m + 0 = m, \forall\, m \in \mathbb{N}$.
(xiii) 1 is the unique element such that $1 \cdot m = m \cdot 1 = m, \forall\, m \in \mathbb{N}$.
(xiv) For each element $[x,y] \in \mathbb{Z}$, $[y,x] \in \mathbb{Z}$ is the unique element such that

$$[x,y] + [y,x] = 0.$$

(xv) Let $a, b, c \in \mathbb{Z}$**.**Then $c = a - b \Leftrightarrow a = c + b$.
(xvi) For all $b \in \mathbb{Z}$**,** $-(-b) = b$.
Proof. Exercise.

Remark 2.2.10.
For each element $a = [x,y] \in \mathbb{Z}$**,** the unique element in Theorem 2.2.8(xiv) is $-a = [y,x]$**.**

Definition 2.2.9. ($\mathbb{Z}$ as an Ordered)
Let $[r,s], [t,u] \in \mathbb{Z}$. We say that $[r,s]$ **less than** $[t,u]$ and denoted by

$$[r,s] < [t,u] \Leftrightarrow r + u < s + t.$$

This is well defined and agrees with the ordering on N.

Theorem 2.2.10.(Trichotomy For $\mathbb{Z}$)
For each $[r,s], [t,u] \in \mathbb{Z}$ one and only one of the following is true:
(1) $[r,s] < [t,u]$ or **(2)** $[t,u] < [r,s]$ or **(3)** $[r,s] = [t,u]$.
Proof.
Since $r + u, t + s \in \mathbb{N}$, so by Trichotomy Theorem for $\mathbb{N}$ one and only one of the following is true:
(1) $r + u < s + t \longrightarrow [r,s] < [t,u]$
(2) $s + t < r + u \longrightarrow [t,u] < [r,s]$
(3) $r + u = s + t \longrightarrow (r,s)R^*(t,u) \longrightarrow [r,s] = [t,u]$

Theorem 2.2.11.
For each $[r,s] \in \mathbb{Z}$, $[r,s] < [0,0] \Leftrightarrow r < s$.

Proof.
$[r,s] < [0,0] \Leftrightarrow r + 0 < s + 0 \Leftrightarrow r < s$.

Remark 2.2.12.
According to Theorem 2.2.11 and Notation 2.2.7(i), for all $[r,s] \in \mathbb{Z}$
$[r,s] < [0,0] \Leftrightarrow r < s \Leftrightarrow [r, r + l]$, where $s = r + l \Leftrightarrow [0, l] < [0,0]$
$\Leftrightarrow -l < 0$.