Foundation of Mathematics II

Chapter 3 Rational Numbers and Groups

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1. Construction of Rational Numbers

Consider the set

$$V = \{(r,s) \in \mathbb{Z} \times \mathbb{Z} \mid r,s \in Z, s \neq 0\}$$

of pairs of integers. Let us define an equivalence relation on V by putting

$$\boxed{(r,s)\ L^*\ (t,u) \Leftrightarrow ru = st}.$$

This is an equivalence relation. **(Exercise).**

Let

$$[r,s] = \{(x,y) \in V \mid (x,y)\ L^*\ (r,s)\},$$

denote the equivalence class of (r,s) and write $[r,s] = \frac{r}{s}$. Such an equivalence class $[r,s]$ is called a **rational number.**

Example 3.1.1.
$(2,12)\ L^*\ (1,6)$ since $2 \cdot 6 = 12 \cdot 1$,
$(2,12)\ \cancel{L^*}\ (1,7)$ since $2 \cdot 7 \neq 12 \cdot 1$.
$[0,1] = \{(x,y) \in V | 0y = x1\} = \{(x,y) \in V | 0 = x\} = \{(0,y) \in V | y \in \mathbb{Z}\}$
$= \{(0,\pm 1),(0,\pm 2),\dots\}$.

Definition 3.1.2. (Rational Numbers)
The set of all equivalence classes $[r,s]$ (rational number) with $(r,s) \in V$ is called the **set of rational numbers** and denoted by $\mathbb{Q}$.

3.1. 3. Addition and Multiplication on $\mathbb{Q}$
Addition: $\oplus: \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$**;**

$$\boxed{[r,s] \oplus [t,u] = [ru + ts, s\,u]},\ s,u \neq 0$$

Multiplication: $\odot: \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$**;**

$$\boxed{[r,s] \odot [t,u] = [rt, su]}\ s,u \neq 0$$

Remark 3.1.4. The relation $i: \mathbb{Z} \longrightarrow \mathbb{Q}$, defined by $i(n) = [n,1]$ is 1-1 function, and

$$i(n+m) = i(n) \oplus i(m),$$
$$i(n \cdot m) = i(n) \odot i(m).$$

Theorem 3.1.5.

(i) $n \oplus m = m \oplus n$, $\forall\, n, m \in \mathbb{Q}$. (Commutative property of $\oplus$)

(ii) $(n \oplus m) \oplus c = n \oplus (m \oplus c)$, $\forall\, n, m, c \in \mathbb{Q}$. (Associative property of $\oplus$)

(iii) $n \odot m = m \odot n$, $\forall\, n, m \in \mathbb{Q}$. (Commutative property of $\odot$)

(iv) $(n \odot m) \odot c = n \odot (m \odot c)$, $\forall\, n, m, c \in \mathbb{Q}$. (Associative property of $\odot$)

(v) $(n \oplus m) \odot c = (n \odot c) \oplus (m \odot c)$ (Distributive law of $\odot$ on $\oplus$)

(vi) (Cancellation Law for $\oplus$).

$$m \oplus c = n \oplus c, \text{ for some } c \in \mathbb{Q} \Leftrightarrow m = n.$$

(vii) (Cancellation Law for $\odot$).

$$m \odot c = n \odot c, \text{ for some } c (\neq 0) \in \mathbb{Q} \Leftrightarrow m = n.$$

(viii) $[0,1]$ is the unique element such that $[0,1] \oplus m = m \oplus [0,1] = m$, $\forall\, m \in \mathbb{Q}$.

(ix) $[1,1]$ is the unique element such that $[1,1] \odot m = m \odot [1,1] = m$, $\forall\, m \in \mathbb{Q}$.

Proof.

(vi) Let $m = [m_1, m_2]$, $n = [n_1, n_2]$, $c = [c_1, c_2] \in \mathbb{Q}$, m_i, n_i, $c_i \in \mathbb{Z}$, $i = 1,2$.

$m \oplus c = n \oplus c$

$\leftrightarrow [m_1, m_2] \oplus [c_1, c_2] = [n_1, n_2] \oplus [c_1, c_2]$

$\leftrightarrow [m_1c_2 + c_1m_2, m_2c_2] = [n_1c_2 + c_1n_2, n_2c_2]$ Def. of $\oplus$ for $\mathbb{Q}$

$\leftrightarrow (m_1c_2 + c_1m_2, m_2c_2) L^* (n_1c_2 + c_1n_2, n_2c_2)$ Def. of $[a, b]$

$\leftrightarrow (m_1c_2 + c_1m_2) n_2c_2 = (n_1c_2 + c_1n_2) m_2c_2$ Def. of L^*

$\leftrightarrow ((m_1n_2)c_2 + (n_2m_2)c_1)c_2 = ((n_1m_2)c_2 + (n_2\, m_2)c_1)c_2$ Properties of $+$ and $\cdot$ on $\mathbb{Z}$

$\leftrightarrow (m_1n_2)c_2 + (n_2m_2)c_1 = (n_1m_2)c_2 + (n_2\, m_2)c_1$ Cancel. law for $\cdot$

$\leftrightarrow (m_1n_2)c_2 = (n_1m_2)c_2$ Cancel. law for $+$

$\leftrightarrow (m_1n_2) = (n_1m_2)$ Cancel. law for $\cdot$

$\leftrightarrow (m_1n_2) L^* (n_1m_2)$ Def. of L^*

$\leftrightarrow [m_1, m_2] = [n_1, n_2]$ Def. of $[a, b]$

(vii) Let $m = [m_1, m_2]$, $n = [n_1, n_2]$, $c = [c_1, c_2] \in \mathbb{Q}$, m_i, n_i, $c_i (\neq [0,1]) \in \mathbb{Z}$, $i = 1,2$.

$m \odot c = n \odot c$

$\leftrightarrow [m_1, m_2] \odot [c_1, c_2] = [n_1, n_2] \odot [c_1, c_2]$

$\leftrightarrow [m_1c_1, m_2c_2] = [n_1c_1, n_2c_2]$ Def. of $\odot$ for $\mathbb{Q}$

$\leftrightarrow (m_1c_1, m_2c_2) L^* (n_1c_1, n_2c_2)$ Def. of $[a, b]$

$\leftrightarrow (m_1c_1)(n_2c_2) = (n_1c_1)(m_2c_2)$ Def. of L^*

$\leftrightarrow (m_1n_2)(c_1c_2) = (m_2n_1)(c_1c_2)$ Asso. and comm.. of $+$ and $\cdot$ on $\mathbb{Z}$

$\leftrightarrow (m_1n_2) = (m_2n_1)$ Cancel. law for $\cdot$

$\leftrightarrow\ (m_1 n_2)L^*(n_1 m_2)$ Def. of L^*
$\leftrightarrow [m_1, m_2] = [n_1, n_2]$ Def. of $[a, b]$

(i),(ii),(iii),(iv)(v),(viii),(ix) Exercise.

Definition 3.1.6.
(i) An element $[n, m] \in \mathbb{Q}$ is said to be **positive element if** $nm > 0$. The set of all positive elements of $\mathbb{Q}$ will denoted by $\mathbb{Q}^+$**.**
(ii) An element $[n, m] \in \mathbb{Q}$ is said to be **negative element if** $nm < 0$. The set of all positive elements of $\mathbb{Q}$ will denoted by $\mathbb{Q}^-$**.**

Remark 3.1.7. Let $[r, s]$ be any rational number. If $s < -1$ or $s = -1$ we can rewrite this number as $[-r, -s]$; that is, $[r, s] = [-r, -s]$.

Definition 3.1.8. Let $[r, s], [t, u] \in \mathbb{Q}$. We say that $[r, s]$ **less than** $[t, u]$ and denoted by

$$[r, s] < [t, u] \Leftrightarrow ru < st,$$

where $s, u > 1$ or $s, u = 1$.

Example 3.1.9.
$[2, 5], [7, -4] \in \mathbb{Q}$**.**
$[2,5] \in \mathbb{Q}^+$**,** since $2 = [2,0]$, $5 = [5,0]$ in $\mathbb{Z}$ and $2 \cdot 5 = [2 \cdot 5 + 0 \cdot 0, 2 \cdot 0 + 5 \cdot 0]$
$= [10,0] = +10 > 0.$
$[-4,7] \in \mathbb{Q}^-$**,** since $7 = [7,0]$, $-4 = [0,4]$ in $\mathbb{Z}$ and

$$7 \cdot (-4) = [7 \cdot 0 + 0 \cdot 4, 7 \cdot 4 + 0 \cdot 0]$$
$$= [0,32] = -32 < 0.$$

$[-4, 7] < [2,5]$, since $-4 \cdot 5 < 2 \cdot 7$.
$[7, -4] < [2,5]$, since $[7, -4] = [-7, -(-4)] = [-7,4]$, and $-7 \cdot 5 < 2 \cdot 4$.

2. Binary Operation

Definition 3.2.1. Let A be a non empty set. The relation $*: A \times A \to A$ is called a (**closure) binary operation** if $\boxed{*(a,b) = a * b \in A,\ \forall a, b \in A}$; that is, $*$ is function.

Definition 3.2.2. Let A be a non empty set and $*,\cdot$ be binary operations on A. The pair $(A,*)$ is called **mathematical system with one operation,** and the triple $(A,*,\cdot)$ is called **mathematical system with two operations.**

Definition 3.2.3. Let $*$ and $\cdot$ be binary operations on a set A.

(i) $*$ is called **commutative** if $\boxed{a * b = b * a, \forall a, b \in A}$.

(ii) $*$ is called **associative** if $\boxed{(a * b) * c = a * (b * c), \forall a, b, c \in A}$.

(iii) $\cdot$ is called **left distributive over** $*$ if

$$\boxed{(a * b) \cdot c = (a \cdot c) * (b \cdot c), \forall a, b, c \in A}.$$

(iv) $\cdot$ is called **right distributive over** $*$ if

$$\boxed{a \cdot (b * c) = (a \cdot b) * (a \cdot c), \forall a, b, c \in A}.$$

Definition 3.2.4. Let $*$ be a binary operation on a set A.

(i) An element $\boldsymbol{e} \in A$ is called an **identity with respect to** $*$ if

$$\boxed{a * e = e * a = a, \forall a \in A}.$$

(ii) If A has an identity element $\boldsymbol{e}$ with respect to $*$ and $a \in A$, then an element b of A is said to be an **inverse of $\boldsymbol{a}$ with respect to** $*$ if

$$\boxed{a * b = b * a = e}.$$

Example 3.2.5. Let X be a non empty set.

(i) $(P(X), \cup)$ formed a mathematical system with identity $\emptyset$.

(ii) $(P(X), \cap)$ formed a mathematical system with identity X.

(iii) $(\mathbb{N}, +)$ formed a mathematical system with identity 0.

(iv) $(\mathbb{Z}, +)$ formed a mathematical system with identity 0 and $-a$ an inverse for every $a(\neq 0) \in \mathbb{Z}$.

(iv) $(\mathbb{Z}\backslash\{0\}, \cdot)$ formed a mathematical system with identity 1.

Theorem 3.2.6. Let $*$ be a binary operation on a set A.

(i) If A has an identity element with respect to $*$, then this identity is unique.

(ii) Suppose A has an identity element $\boldsymbol{e}$ with respect to $*$ and $*$ is associative. Then the inverse of any element in A if exist it is unique.

Proof.

(i) Suppose $\boldsymbol{e}$ and $\hat{\boldsymbol{e}}$ are both identity elements of A with respect to $*$.

(1) $a * \boldsymbol{e} = \boldsymbol{e} * a = a, \forall a \in A$ (Def. of identity)

And

(2) $a * \hat{\boldsymbol{e}} = \hat{\boldsymbol{e}} * a = a, \forall a \in A$ (Def. of identity)

(3) $\hat{\boldsymbol{e}} * e = e * \hat{\boldsymbol{e}} = \hat{\boldsymbol{e}}$ (Since (1) is hold for $a = \hat{\boldsymbol{e}}$)

(4) $e * \hat{\boldsymbol{e}} = \hat{\boldsymbol{e}} * e = \boldsymbol{e}$ (Since (2) is hold for $a = \boldsymbol{e}$)

(5) $\boldsymbol{e} = \hat{\boldsymbol{e}}$ (Inf. (3) and (4))

(ii) Let $a \in A$ has two inverse elements say b and c with respect to $*$. To prove $b = c$.

(1) $a * b = b * a = e$ (Def. of inverse (b inverse element of a))

(2) $a * c = c * a = e$ (Def. of inverse (c inverse element of a))

(3) $b = b * e$ (Def. of identity)

$= b * (a * c)$ (From (2) Rep(e: $a * c$))

$= (b * a) * c$ (Since $*$ is associative)

$= e * c$ (From (i) Rep($b * a : e$)) and

$= c$ (Def. of identity).

Therefore; $b = c$.

Definition 3.2.7. A mathematical system with one operation, $(G,*)$ is said to be

(i) semi group if $\boxed{(a * b) * c = a * (b * c), \forall a, b, c \in G}$. (Associative law)

(ii) group if

(1) (Associative law) $\boxed{(a * b) * c = a * (b * c), \forall a, b, c \in G}$.

(2) (**Identity with respect to** $*$) There exist an element e such that $a * e = e * a = a, \forall a \in A$.

(3) (**Inverse with respect to** $*$) For all $a \in G$, there exist an element $b \in G$ such that $\boxed{a * b = b * a = e}$.

(4) If the operation $*$ is commutative on G then the group is called **commutative group;** that is, $\boxed{a * b = b * a, \forall a, b \in G}$.

Example 3.2.8. (i) Let G be a non empty set. $(P(G), \cup)$ and $(P(G), \cap)$ are not group since it has no inverse elements, but they are semi groups.

(ii) $(\mathbb{N}, +)$, $(\mathbb{N},\cdot)$ and $(\mathbb{Z},\cdot)$, are not groups since they have no inverse elements, but they are semi groups.

(iii) $(\mathbb{Z}, +)$, $(\mathbb{Q}\backslash\{0\},\cdot)$, are commutative groups.

Symmetric Group 3.2.9.

Let $X = \{1,2,3\}$, and S_3 =Set of All permutations of 3 elements of the set X.

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3	2	1

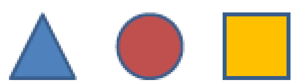
There are 6 possiblities and all possible permutations of X as follows:

1			2			3			4			5			6		
1	2	3	1	3	2	2	1	3	2	3	1	3	1	2	3	2	1

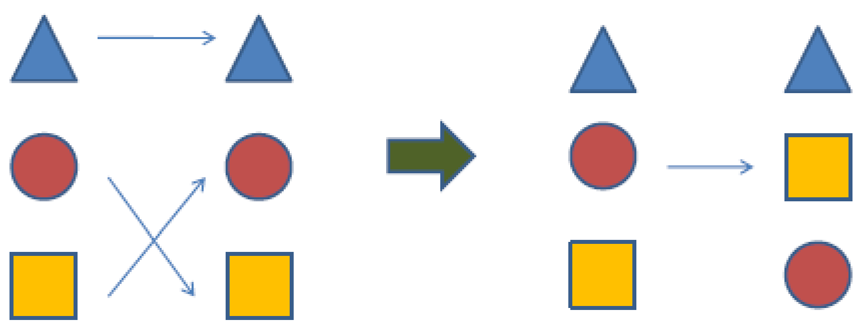
Let $\sigma_i : X \to X$, $i = 1,2, \dots 6$, defined as follows:

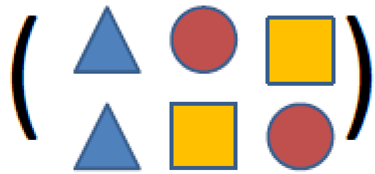
$\sigma_1(1) = 1$ $\sigma_1(2) = 2$ $\sigma_1(3) = 3$	$\sigma_2(1) = 2$ $\sigma_2(2) = 1$ $\sigma_2(3) = 3$	$\sigma_3(1) = 3$ $\sigma_3(2) = 2$ $\sigma_3(3) = 1$
$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = ()$	$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)$	$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)$
$\sigma_4(1) = 1$ $\sigma_4(2) = 3$ $\sigma_4(3) = 2$	$\sigma_5(1) = 2$ $\sigma_5(2) = 3$ $\sigma_5(3) = 1$	$\sigma_6(1) = 3$ $\sigma_6(2) = 1$ $\sigma_6(3) = 2$
$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23)$	$\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$	$\sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$

$S_3 = \{\sigma_1 = (\quad) = e, \sigma_2 = (12), \sigma_3 = (13), \sigma_4 = (23) =, \sigma_5 = (123), \sigma_6 = (132)\}.$

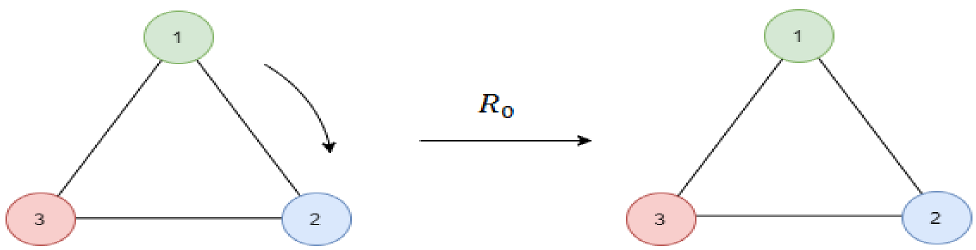
- X={  }

- Define an arbitrary bijection

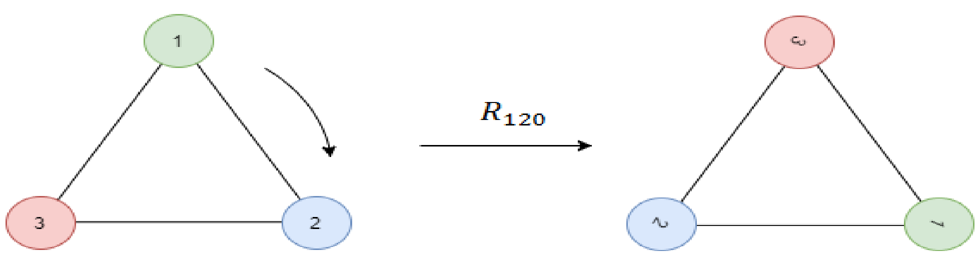




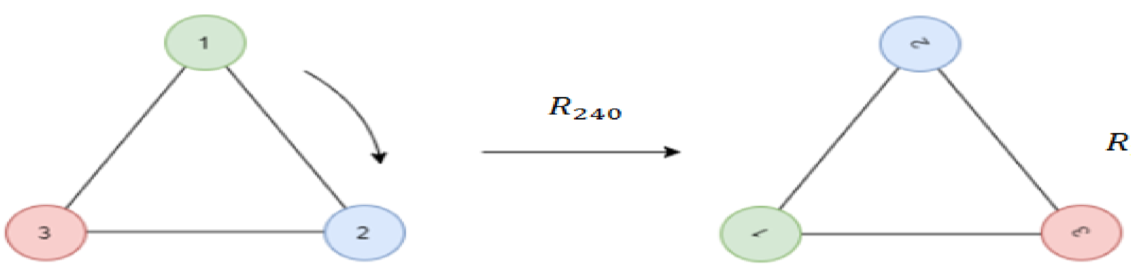
$$\sigma_4 = (23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



$R_0 = \sigma_1$



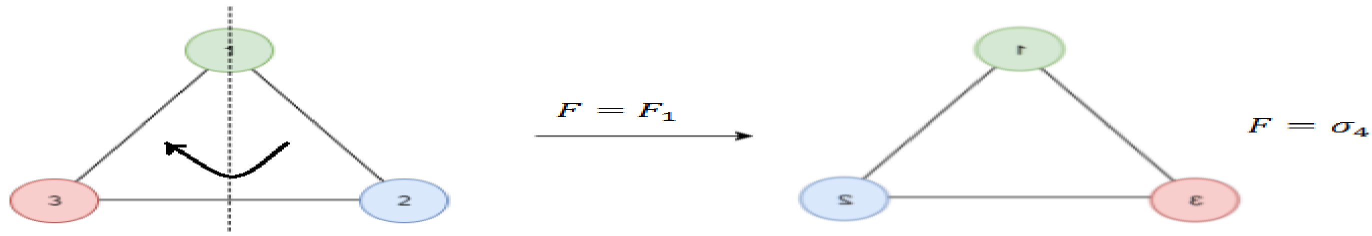
$R_{120} = \sigma_5$



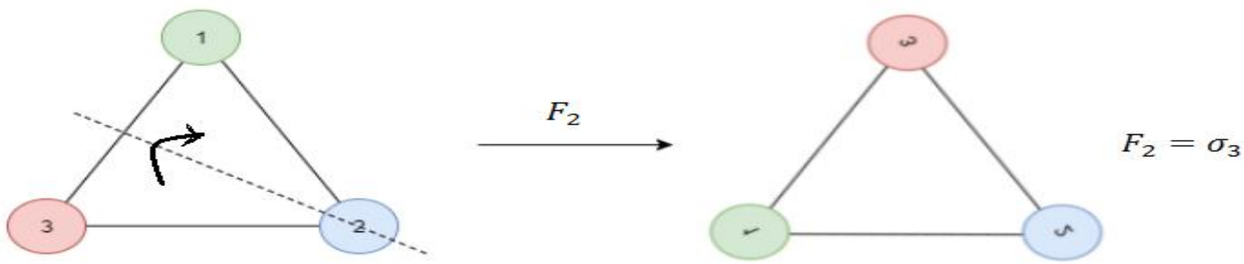
$R_{240} = \sigma_6$

Note that $R_{240} = R_{120} \circ R_{120} = {R_{120}}^2$.
Draw a vertical line through the top corner $\boxed{i}$, $i = 1,2,3$ and flip about this line.

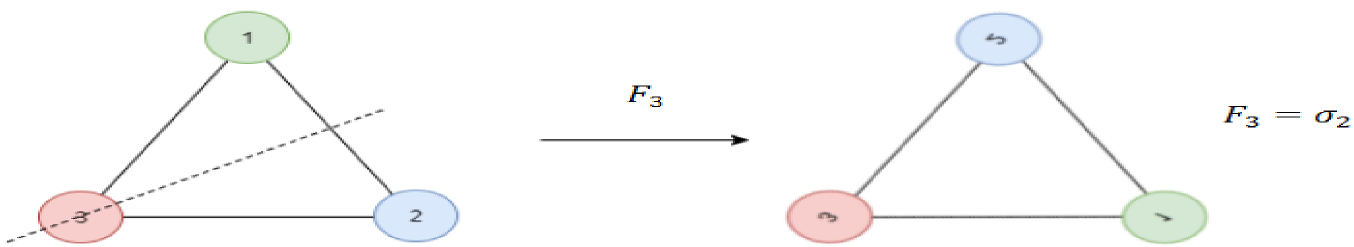
1- If $i = 1$ call this operation $F = F_1$.



2- If $i = 2$ call this operation F_1.



3- If $i = 3$ call this operation F_3.



Note that $F^2 = F \circ F = \sigma_1$, representing the fact that flipping twice does nothing.

❖ All permutations of a set X of 3 elements form a group under composition $\circ$ of functions, called the **symmetric group** on 3 elements, denoted by S_3. (Composition of two bijections is a bijection).

$\circ$	$\boldsymbol{\sigma_1 = e}$	$\boldsymbol{\sigma_2 = (12)}$	$\boldsymbol{\sigma_3 = (13)}$	$\boldsymbol{\sigma_4 = (23)}$	$\boldsymbol{\sigma_5 = (123)}$	$\boldsymbol{\sigma_6 = (132)}$
$\boldsymbol{\sigma_1 = e}$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
$\boldsymbol{\sigma_2 = (12)}$	σ_2	e	σ_6	σ_5	σ_4	σ_3
$\boldsymbol{\sigma_3 = (13)}$	σ_3	σ_5	e	σ_6	σ_2	σ_4
$\boldsymbol{\sigma_4 = (23)}$	σ_4	σ_6	σ_5	e	σ_3	σ_2
$\boldsymbol{\sigma_5 = (123)}$	σ_5	σ_3	σ_4	σ_2	σ_6	e
$\boldsymbol{\sigma_6 = (132)}$	σ_6	σ_4	σ_2	σ_3	e	σ_5

$\sigma_2 \circ \sigma_3$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$\sigma_5 \circ \sigma_2$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$