Alternating Current Circuits

5.1. AC Sources

An AC circuit consists of circuit elements and a power source that provides an alternating voltage Δv. This time-varying voltage is described by:

$$\boldsymbol{\Delta v = \Delta V_{max} sin\omega t}$$

where $\boldsymbol{\Delta V_{max}}$ is the maximum output voltage of the AC source, or the voltage amplitude. There are various possibilities for AC sources, including generators and electrical oscillators. In a home, each electrical outlet serves as an AC source.

The units of are cycles per second $\boldsymbol{f = \frac{1}{T} = \frac{\omega}{2\pi}}$

$$\boldsymbol{\omega = 2\pi f}$$

Where f is the frequency of the source and T is the period. The source determines the frequency of the current in any circuit connected to it.

5.2 Resistors in an AC Circuit

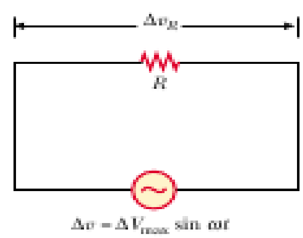


Figure 5.1

Consider a simple AC circuit consisting of a resistor and an AC source, as shown in Fig. 5.1. At any instant, the algebraic sum of the voltages around a closed loop in a circuit (Kirchhoff's loop rule). Therefore,

$$\boldsymbol{\Delta v + \Delta v_R = 0}$$

So that the magnitude of the source voltage equals the magnitude of the voltage across the resistor:

$$\boldsymbol{\Delta v = \Delta v_R = \Delta V_{max} sin\omega t} \qquad ----- \boldsymbol{1}$$

Where $\boldsymbol{\Delta v_R}$ is the instantaneous voltage across the resistor. And $R = \Delta V / I$, the instantaneous current in the resistor is:

$$i_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{max} sin\omega t}{R} = I_{max}\ sin\omega t$$

Where I max is the maximum current:

$I_{max} = \frac{\Delta V_{max} sin\omega t}{R}$ Substituted in eq.1 we get

$$\Delta v_R = I_{max}\ R\ sin\omega t$$

5.3 Inductors in an AC Circuit

Consider an AC circuit consisting only of an inductor connected to the terminals of an AC source, as shown in Fig.5.2. If $\Delta v_L = \varepsilon_L = -L\frac{dI}{dt}$ is the self-induced instantaneous voltage across the inductor, then Kirchhoff's loop rule applied to this circuit gives $\Delta v + \Delta v_L = 0$, or

$$\Delta v - L\frac{dI}{dt} = 0$$

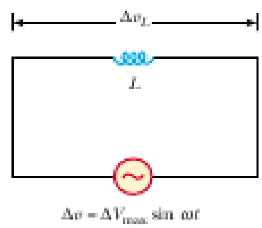


Figure 5.2

When we substitute $\Delta v = \Delta V_{max} sin\omega t$ we obtain:

$$\Delta V_{max} sin\omega t\ - L\frac{dI}{dt} = 0$$

$$\Delta V_{max} sin\omega t\ = L\frac{dI}{dt}$$

Solving this equation for dI, we find that:

$$dI\ = \frac{\Delta V_{max}\ sin\omega t\ dt}{L}$$

Integrating this expression gives the instantaneous current I_L in the inductor as a function of time:

$$I_L = \frac{\Delta V_{max}}{L}\int sin\omega t\ dt = -\frac{\Delta V_{max}}{\omega L} cos\omega t$$

5.4 Capacitors in an AC Circuit

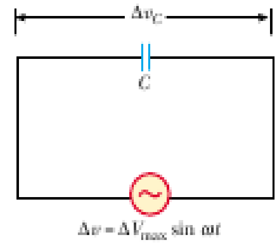


Figure 5.3

Fig.5.3 shows an AC circuit consisting of a capacitor connected across the terminals of an AC source. Kirchhoff's loop rule applied to this circuit gives,

$\Delta v + \Delta v_C = 0$

so that the magnitude of the source voltage is equal to the magnitude of the voltage across the capacitor:

$$\Delta v = \Delta v_C = \Delta V_{max} sin\omega t \qquad ----1$$

Where Δv_C is the instantaneous voltage across the capacitor. We know from the definition of capacitance that $= q/\Delta v_C$:

$$q = C\, \Delta V_{max}\, sin\omega t \qquad ----2$$

Where q is the instantaneous charge on the capacitor. Because $I_C = dq/dt$, differentiating Eq.2with respect to time gives the instantaneous current in the circuit:

$$I_C = \frac{dq}{dt} = \omega\, C\, \Delta V_{max}\, sin\omega t$$

Using the trigonometric identity

$$cos\omega t = \sin(\omega t + \frac{\pi}{2})$$

$$I_C = \frac{dq}{dt} = \omega\, C\, \Delta V_{max} \sin\left(\omega t + \frac{\pi}{2}\right) \qquad ----3$$

for a sinusoidally applied voltage, the current always leads the voltage across a capacitor by 90°.

From Eq. 3, we see that the current in the circuit reaches its maximum value when $cos\omega t = 1$:

$$I_{max} = \omega\, C\, \Delta V_{max} = \frac{\Delta V_{max}}{(1/\omega C)} \qquad ----4$$

We give the **capacitive reactance** $X_C = 1/\omega C$, because this function varies with frequency. We can write Eq. 4 as:

$$I_{max} = \frac{\Delta V_{max}}{X_C} \qquad ----5$$

Combining Eq. 5 and 1, we can express the instantaneous voltage across the capacitor as:

$$\Delta v = \Delta v_C = \Delta V_{max} sin\omega t = I_{max}\, X_C\, sin\omega t$$

Example: An **8μF** capacitor is connected to the terminals of a **60Hz** AC source whose **rms** voltage is **150 V**. Find the capacitive reactance and the **rms** current in the circuit.

Solution:

$$\omega = 2\pi f = 2\pi \times 60Hz = 377sec^{-1}$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(377sec^{-1})(8\times 10^{-6}F)} = 332\Omega$$

$$I_{rms} = \frac{\Delta V_{rms}}{X_C} = \frac{150V}{332\Omega} = 0.452A$$

5.5. The RLC Series Circuit and Resonance

Figure 5.4 shows a circuit that contains a resistor, an inductor, and a capacitor connected in series across an alternating voltage source. As before, we assume that the applied voltage varies sinusoidally with time. It is convenient to assume that the instantaneous applied voltage is given by:

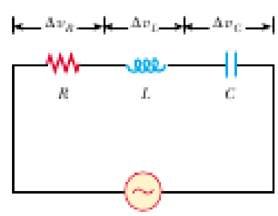


Figure 5.4

$$\Delta v = \Delta V_{max} sin\omega t$$

While the current varies as

$$i = I_{max} sin(\omega t - \emptyset)$$

Where ∅ is some phase angle between the current and the applied voltage. The current at all points in a series AC circuit has the same amplitude and phase. We can express the instantaneous voltages across the three circuit elements as:

$$\Delta_{VR} = \boldsymbol{I_{max} R \, sin\omega t} = \Delta \boldsymbol{V_R \, sin\omega t}$$

$$\Delta_{VL} = \boldsymbol{I_{max} X_L \sin(\omega t + \frac{\pi}{2})} = \Delta \boldsymbol{V_L \, cos\omega t}$$

$$\Delta_{VC} = \boldsymbol{I_{max} X_C \sin\left(\omega t - \frac{\pi}{2}\right)} = -\Delta \boldsymbol{V_C \, cos\omega t}$$

Where Δ_{VR}, Δ_{VL} and Δ_{VC} are the maximum voltage values across the elements: $\Delta_{VR} = \boldsymbol{I_{max} R}$, $\Delta_{VL} = \boldsymbol{I_{max} X_L}$ **and** $\Delta_{VC} = \boldsymbol{I_{max} X_C}$

At this point, we could proceed by noting that the instantaneous voltage $\Delta\nu$ across the three elements equals the sum:

$$\Delta\nu = \Delta_{VR} + \Delta_{VL} + \Delta_{VC}$$

In the Figure 5.5 from this diagram, we see that the vector sum of the voltage amplitudes Δ_{VR} , Δ_{VL} and Δ_{VC} equals a phasor whose length is the maximum applied voltage $\Delta\nu_{max}$ and which makes an angle ∅ with the current phasor $\boldsymbol{I_{max}}$. The voltage phasors Δ_{VL} and Δ_{VC} are in opposite directions along the same line, so we can construct the difference phasor $\Delta_{VL} - \Delta_{VC}$, which is perpendicular to the phasor Δ_{VR}. From either one of the right triangles:

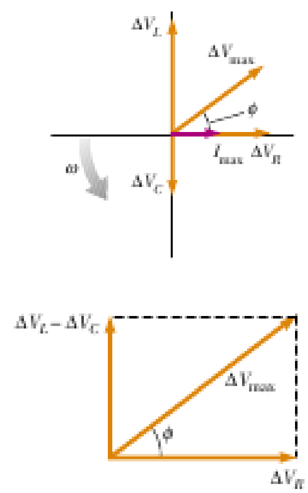


Figure 5.5

$$\Delta V_{max} = \sqrt{{\Delta_{VR}}^2 + (\Delta_{VL} - \Delta_{VC})^2} = \sqrt{(\boldsymbol{I_{mxa} R})^2 + (\boldsymbol{I_{max} X_L} - \boldsymbol{I_{max} X_C})^2}$$

$$\Delta V_{max} = I_{max} \sqrt{\boldsymbol{R^2} + (\boldsymbol{X_L} - \boldsymbol{X_C})^2}$$

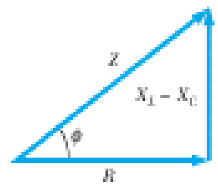
Therefore, we can express the maximum current as:

$$I_{max} = \frac{\Delta V_{max}}{\sqrt{\boldsymbol{R^2 + (X_L - X_C)^2}}}$$

Where $\boldsymbol{Z = \sqrt{R^2 + (X_L - X_C)^2}}$ is called the **impedance**.

$$\Delta \boldsymbol{V_{max} = I_{max}\, Z}$$

find that the phase angle $\emptyset$ between the current and the voltage is:



$$\boldsymbol{tan}\emptyset = \left(\frac{\boldsymbol{X_L - X_C}}{\boldsymbol{R}}\right)$$

The resonance condition for the **Series RLC** circuit is given by $\emptyset = 0$, which implies: $\boldsymbol{X_L = X_C}$ from which we obtain $\boldsymbol{\omega_o L = 1/\omega_o C}$ **:**

The **resonant frequency** is: $\boldsymbol{\omega_o = \frac{1}{\sqrt{LC}}}$

Example: A series **RLC AC** circuit has ***R=425Ω, L=1.25 H, C = 3.5μF,*** $\boldsymbol{\omega = 377 sec^{-1}}$, and $\boldsymbol{V_{max}}$ =**150V**. **(A)** Determine the inductive reactance, the capacitive reactance, and the impedance of the circuit. **(B)** Find the maximum current in the circuit. **(C)** Find the phase angle between the current and voltage. **(D)** Find both the maximum voltage and the instantaneous voltage across each element.

Solution: The reactances are $X_L = \omega L = 471\Omega$ and $X_C = 1/\omega C = 758\Omega$.

The impedance is:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(425\Omega)^2 + (471\Omega - 758\Omega)^2} = 513\Omega$$

$$\varphi = tan^{-1}\left(\frac{X_L - X_C}{R}\right) = tan^{-1}\frac{(471\Omega - 758\Omega)}{425\Omega} = -34°$$

$$\Delta V_R = I_{max}\, R = (0.292A)(425\Omega) = 124V$$

$$\Delta V_L = I_{max}\, X_L = (0.292A)(471\Omega) = 138V$$

$$\Delta V_C = I_{max}\, X_C = (0.292A)(758\Omega) = 221V$$

$\Delta v_R = (124\ V)\ sin377t$

$\Delta v_L = (138\ V)\ cos377t$

$\Delta v_C = (-221\ V)\ cos377t$

Example: Consider a series **RLC** circuit for which **R=150Ω**, **L=20mH**, **ΔV$_{rms}$=20V**, and $= \mathbf{5000 sec^{-1}}$. Determine the value of the capacitance for which the current is a maximum.

Solution:

$$\boldsymbol{\omega_o = \frac{1}{\sqrt{LC}}}$$

$$\boldsymbol{C = \frac{1}{\omega_o^2 L} = \frac{1}{(5000 sec^{-1})^2 (20 \times 10^{-3} H)} = 2\mu F}$$

5.6. Parallel RLC Circuit and Resonance

Consider the parallel RLC circuit illustrated in Fig.5.6. The AC voltage source is:

$$\boldsymbol{\Delta v = \Delta V_{max} sin\omega t}$$

Unlike the series RLC circuit, the instantaneous voltages across all three circuit elements R, L and C are the same, and each voltage is in phase with the current through the resistor. However, the currents through each element will be different. The current in the resistor is:

$$\boldsymbol{I_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{max} sin\omega t}{R} = I_{max}\ sin\omega t}$$

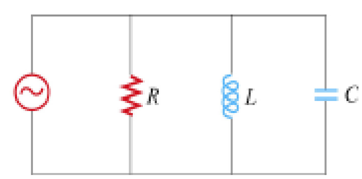


Figure 5.6

The voltage across the inductor is:

$$\boldsymbol{\Delta v_L = \Delta V_{max} sin\omega t = L\frac{dI}{dt}}$$

$$\int_0^{I_L} dI = \int_0^t \frac{V_{max}}{L} sin\omega t\ dt = -\frac{V_{max}}{\omega L} cos\omega t$$

$$I_L = \frac{V_{max}}{X_L} sin(\omega t - \frac{\pi}{2}) = I_{max} sin(\omega t - \frac{\pi}{2})$$

The voltage across the capacitor is $\Delta v_C = \Delta V_{max} sin\omega t$, which implies

$$I_C = \frac{dq}{dt} = \omega\, C\, \Delta V_{max}\, sin\omega t = \frac{\Delta V_{max}}{X_c}\, sin\left(\omega t + \frac{\pi}{2}\right)$$

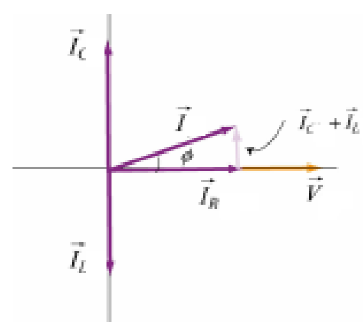
$$I_C = I_{max} sin\left(\omega t + \frac{\pi}{2}\right)$$

Using Kirchhoff's rule, the total current in the circuit is simply the sum of all three currents:

$$I = I_R + I_L + I_C$$

$$I = I_{max}\, sin\omega t + I_{max} \sin(\omega t - \frac{\pi}{2}) + I_{max} sin\left(\omega t + \frac{\pi}{2}\right)$$

From the phasor diagram the maximum amplitude of the total current can be obtained as:



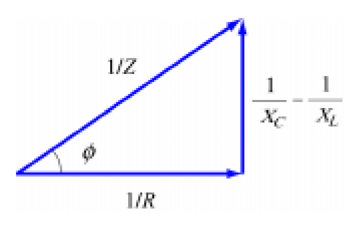
$$I = \sqrt{I_R^2 + (I_L - I_C)^2}$$

$$I = V_{max} \sqrt{\frac{1}{R^2} + (\omega C - \frac{1}{\omega L})^2}$$

$$I = V_{max} \sqrt{\frac{1}{R^2} + (\frac{1}{X_C} - \frac{1}{X_L})^2}$$

$$I = \frac{V_{max}}{Z}$$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + (\frac{1}{X_C} - \frac{1}{X_L})^2}$$



$$\tan\emptyset = \frac{(I_C - I_L)}{I_R} = \frac{(\frac{V_{max}}{X_C} - \frac{V_{max}}{X_L})}{\frac{V_{max}}{R}} = R(\frac{1}{X_C} - \frac{1}{X_L})$$

The resonance condition for the **parallel RLC** circuit is given by $\emptyset = 0$, which implies:

$$\frac{1}{X_C} = \frac{1}{X_L}$$

The **resonant frequency** is: $\omega_o = \frac{1}{\sqrt{LC}}$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

5.7. Power in an AC Circuit

For the **RLC**, we can express the instantaneous power ***P*** as:

$$P = i\Delta v = I_{max}\, sin(\omega t - \varphi)\, \Delta V_{max} sin\omega t$$

$$P = I_{max}\, \Delta V_{max}\, sin\omega t\, sin(\omega t - \varphi)$$

$$sin(\omega t - \varphi) = sin\omega t\, cos\, \varphi - cos\omega t\, sin\, \varphi$$

$$P = I_{max}\, \Delta V_{max}\, sin\omega t\, (sin\omega t\, cos\, \varphi - cos\omega t\, sin\, \varphi)$$

$$P = I_{max}\, \Delta V_{max}\, sin^2\omega t\, cos\, \varphi - I_{max}\, \Delta V_{max}\, sin\omega t\, cos\omega t\, sin\, \varphi)$$

The time average of the second term on the right is identically zero because $sin\omega t\, cos\omega t = \frac{1}{2} sin\, 2\omega t$ and the average value of $sin\, 2\omega t$ ***i***s zero and the average value of $sin^2\omega t = 1/2$, Therefore, we can express the average power P_{av} as:

$$P_{av} = \frac{1}{2} I_{max}\, \Delta V_{max}\, cos\varphi$$

It is convenient to express the average power in terms of the rms current and rms voltage defined by $I_{rms} = I_{max}/\sqrt{2}$ and $\Delta V_{rms} = \Delta V_{max}/\sqrt{2}$

$$\boldsymbol{P_{av} = I_{rms}\,\Delta V_{rms}\,cos\varphi} \qquad ------1$$

Where the quantity $\boldsymbol{cos\varphi}$ called the power factor and the maximum voltage across the resistor is given by $\boldsymbol{\Delta V_R = \Delta V_{max} cos\varphi = I_{max} R}$ and $\boldsymbol{cos\varphi = I_{max} R/\Delta V_{max}}$ substituted in eq.1 we get:

$$\boldsymbol{P_{av} = I_{rms}\,\Delta V_{rms}\,cos\varphi = I_{rms}\left(\frac{\Delta V_{max}}{\sqrt{2}}\right)\frac{I_{max} R}{\Delta V_{max}} = I_{rms}\frac{I_{max} R}{\sqrt{2}}}$$

$$\boldsymbol{I_{max} = \sqrt{2}\ I_{rms}}$$

$$\boldsymbol{P_{av} = I_{rms}^2\ R}$$

Example: A series **RLC AC** circuit has ***R=425Ω, L=1.25 H, C = 3.5µF,*** $\boldsymbol{\omega = 377 sec^{-1}}$, and $\boldsymbol{V_{max}}$ =**150V**. Calculate the average power delivered to the series RLC circuit.

Solution:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = \frac{150V}{\sqrt{2}} = 106V$$

$$\Delta I_{rms} = \frac{\Delta I_{max}}{\sqrt{2}} = \frac{0.292A}{\sqrt{2}} = 0.206A$$

$$\boldsymbol{P_{av} = I_{rms}\,\Delta V_{rms}\,cos\varphi}$$

$$\varphi = -34° \quad \Rightarrow \cos -34° = 0.829$$

$$\boldsymbol{P_{av}} = (0.206A)(106V)(0.829) = 18.1W$$