Inductance

4.1. Self-Inductance

Consider a circuit consisting of a switch, a resistor, and a source of emf, as shown in Figure 4.1. When the switch is thrown to its closed position, the current does not immediately jump from zero to its maximum value $\boldsymbol{\varepsilon/R}$. Faraday's law of electromagnetic induction can be used to describe this effect as follows: as the current increases with time, the magnetic flux through the circuit loop due to this current also increases with time. This increasing flux creates an induced **emf** in the circuit. The direction of the induced **emf** is such that it would cause an induced current in the loop (if the loop did not already carry a current), which would establish a magnetic field opposing the change in the original magnetic field. Thus, the direction of the induced **emf** is opposite the direction of the **emf** of the battery; this results in a gradual rather than the instantaneous increase in the current to its final equilibrium value. Because of the direction of the induced **emf**, it is also called a back **emf**. This effect is called self-induction because the changing flux through the circuit and the resultant induced **emf** arise from the circuit itself. The **emf** $\boldsymbol{\varepsilon_L}$ set up in this case is called a **self-induced emf**.

To obtain a quantitative description of self-induction, we recall from Faraday's law that the induced **emf** is equal to the negative of the time rate of change of the magnetic flux. The magnetic flux is proportional to the magnetic field due to the current, which in turn is proportional to the current in the circuit. Therefore, **a self-induced emf is always proportional to the time rate of change of the current**. For any coil, we find that

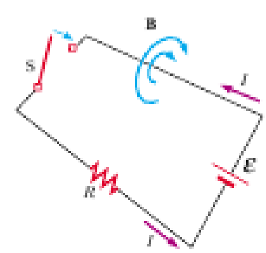


Figure 4.1

$$\boldsymbol{\varepsilon_L = -L\frac{dI}{dt} \qquad ---1}$$

Where **L** is a proportionality constant called the inductance of the coil that depends on the geometry of the coil and other physical characteristics. Combining this expression with Faraday's law, $\boldsymbol{\varepsilon_L = -N\, d\Phi_B/dt}$, we see that the inductance of a closely spaced coil of N turns (a toroid or an ideal solenoid) carrying a current **I** and containing **N** turns is:

$$\boldsymbol{L = N\frac{\Phi_B}{I}}$$

Where it is assumed that the same magnetic flux passes through each turn. From Equation 1, we can also write the inductance as the ratio

$$\boldsymbol{L = -\frac{\varepsilon_L}{dI/dt}}$$

The SI unit of inductance is the **henry (H)**, and $H = \frac{Volt.sec}{Amper}$

Example: (A) Calculate the inductance of an air-core solenoid containing **300 turns** if the length of the solenoid is **25cm** and its cross-sectional area is **4cm^2**. (B) Calculate the **self-induced emf** in the solenoid if the current it carries is decreasing at the rate of **50A/s**.

Solution:

$$\boldsymbol{L = N\frac{\Phi_B}{I}}$$

$$\Phi_{\boldsymbol{B}} = BA = \mu_o \frac{NI}{\ell} A$$

$$\boldsymbol{L} = \mu_o \frac{\boldsymbol{N^2 AI}}{\boldsymbol{I\ell}} = \mu_o \frac{N^2 A}{\ell}$$

$$\boldsymbol{L} = \frac{(4\pi \times 10^{-7} T.m/A)(300)^2 (4 \times 10^{-4} m^2)}{25 \times 10^{-2} m}$$

$$L = 1.81 \times 10^{-4} T.m/A = 1.81 \times 10^{-4} H$$

$$\boldsymbol{\varepsilon_L = -L\frac{dI}{dt}} = -(1.81 \times 10^{-4} H) \times (-50\frac{A}{sec}) = 9.05 mV$$

4.2 RL Circuits

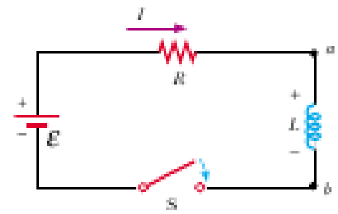


Figure 4.2

Consider the circuit shown in Fig.4.2, which contains a battery of negligible internal resistance. This is an **RL** circuit because the elements connected to the battery are a resistor and an inductor. Suppose that the switch S is open for **t< 0** and then closed at **t=0**. The current in the circuit begins to increase, and a back **emf (Eq.1)** that opposes the increasing current is induced in the inductor. Because the current is increasing, **dI/dt** in **Eq.1** is positive; thus, **L** is negative. This negative value reflects the decrease in electric potential that occurs in going from a to b across the inductor, as indicated by the positive and negative signs in Figure 4.2. With this in mind, we can apply Kirchhoff's loop rule to this circuit, traversing the circuit in the clockwise direction:

$$\boldsymbol{\varepsilon - IR - L\frac{dI}{dt} = 0 \qquad --- \ 2}$$

Where **IR** is the voltage drop across the resistor. (We developed Kirchhoff's rules for circuits with steady currents, but they can also be applied to a circuit in which the current is changing if we imagine them to represent the circuit at one instant of time.) We must now look for a solution to this differential equation, which is similar to that for the **RC** circuit.

A mathematical solution of Eq. 2. represents the current in the circuit as a function of time. To find this solution, we divided Eq.2 on **R**, and letting $\boldsymbol{x = (\varepsilon/R) - I}$, so that $\boldsymbol{dx = -dI}$. With these substitutions, we can write Eq. 2 as:

$$\boldsymbol{x - \frac{L}{R}\frac{dx}{dt} = 0}$$

$$\boldsymbol{\frac{dx}{x} = -\frac{R}{L}\,dt}$$

Integrating this last expression, we have

$$\int_{x_o}^{x} \frac{dx}{x} = -\frac{R}{L} \int_{0}^{t} dt$$

$$\ln \frac{x}{x_o} = -\frac{R}{L}\, t$$

where $\mathbf{x_o}$ is the value of $\mathbf{x}$ at time $\mathbf{t=0}$. Taking the antilogarithm of this result, we obtain:

$$x = x_o\, e^{\frac{-Rt}{L}}$$

Because $\mathbf{I = 0}$ at $\mathbf{t = 0}$, we note from the definition of x that $x = \varepsilon/R$. Hence, this last expression is equivalent to:

$$\frac{\varepsilon}{R} - I = \frac{\varepsilon}{R}\, e^{\frac{-Rt}{L}}$$

$$I = \frac{\varepsilon}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$$

This expression shows how the inductor effects the current. The current does not increase instantly to its final equilibrium value when the switch is closed but instead increases according to an exponential function. If we remove the inductance in the circuit, which we can do by letting **L** approach zero, the exponential term becomes zero and we see that there is no time dependence of the current in this case the current increases instantaneously to its final equilibrium value in the absence of the inductance. We can also write this expression as:

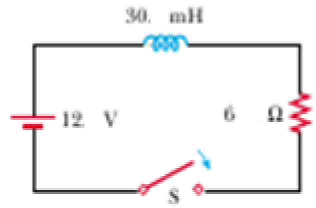
$$I = \frac{\varepsilon}{R}\left(1 - e^{\frac{-t}{\tau}}\right)$$

Where the constant $\boldsymbol{\tau}$ is the time constant of the **RL** circuit: $\tau = L/R$

Physically, τ is the time interval required for the current in the circuit to reach $1 - e^{-1}A = 0.632 = 63.2\%$ of its final value ε/R .

The time constant is a useful parameter for comparing the time responses of various circuits.

Example: **(A)** Find the time constant of the circuit shown in Figure. **(B)** The switch in Figure is closed at **t = 0**. Calculate the current in the circuit at **t = 2ms**.



Solution:

$$A)\qquad \tau = \frac{L}{R} = \frac{30\times 10^{-3}H}{6\Omega} = 5ms$$

$$B)\qquad I = \frac{\varepsilon}{R}\left(1 - e^{\frac{-t}{\tau}}\right) = \frac{12V}{6\Omega}\left(1 - e^{-0.4}\right) = 0.659A$$

4.3 Energy in a Magnetic Field

Because the **emf** induced in an inductor prevents a battery from establishing an instantaneous current, the battery must provide more energy than in a circuit without the inductor. Part of the energy supplied by the battery appears as internal energy in the resistor, while the remaining energy is stored in the magnetic field of the inductor. If we multiply each term in Eq. 2 by **I** and rearrange the expression, we have:

$$\varepsilon - IR - L\frac{dI}{dt} = 0 \qquad ---2 \quad \times I$$

$$I\varepsilon - I^2R - IL\frac{dI}{dt} = 0$$

Recognizing $I\varepsilon$ as the rate at which energy is supplied by the battery and **IR**^{2} as the rate at which energy is delivered to the resistor, we see that **LI(dI/dt)** must represent the rate at which energy is being stored in the inductor. If we let **U** denote the energy stored in the inductor at any time, then we can write the rate **dU/dt** at which energy is stored as:

$$\frac{dU}{dt} = IL\frac{dI}{dt}$$

To find the total energy stored in the inductor, we can rewrite this expression as $dU = I\,L\,dI$ and integrate:

$$dU = \int dU = \int_0^I ILdI = L\int_0^I IdI$$

$$U = \frac{1}{2}I^2L \qquad ----3$$

We can also determine the energy density of a magnetic field. When substitute:

$$L = \mu_o n^2 A\ell \quad \text{and} \quad B = \mu_o nI \quad \Rightarrow I = \frac{B}{\mu_o n} \quad \text{and} \quad n = N/\ell$$

$$U = \frac{1}{2}\mu_o n^2 A\ell(\frac{B}{\mu_o n})^2 = \frac{B^2}{\mu_o n}$$

4.4 Mutual Inductance

Consider the two closely wound coils of wire shown in cross-sectional view in Fig. 4.3. The current I_1 in **coil 1**, which has N_1 turns, creates a magnetic field. Some of the magnetic field lines pass through **coil 2**, which has N_2 turns. The magnetic flux caused by the current in coil 1 and passing through **coil 2** is represented by Φ_{12}. In analogy equation $L = (N_B/I)$,we define the mutual inductance M_{12} of **coil 2** with respect to **coil 1**:

$$M_2 = \frac{N_2\Phi_{12}}{I_1} \qquad ----3$$

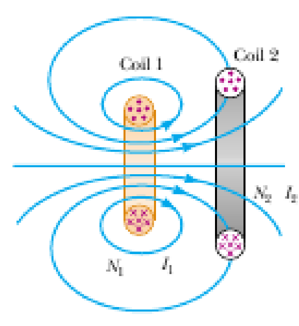


Figure. 4.3

Mutual inductance depends on the geometry of both circuits and on their orientation with respect to each other. As the circuit separation distance increases, the mutual inductance decreases because the flux linking the circuits decreases. If the current I_1 varies with time, we see from Faraday's law and **Eq. 3** that the **emf** induced by **coil 1** in **coil 2** is:

$$\varepsilon_2 = -N_2\frac{d\Phi_{12}}{dt} = -N_2\frac{d}{dt}(\frac{M_{12}I_1}{N_2}) = -M_{12}\frac{dI_1}{dt}$$

If the current $\boldsymbol{I_2}$ varies with time, the **emf** induced by **coil 2** in **coil 1** is

$$\boldsymbol{\varepsilon_1 = -M_{21}\frac{dI_2}{dt}}$$

In mutual induction, the **emf** induced in one coil is always proportional to the rate at which the current in the other coil is changing.

4.5. Oscillations in an LC Circuit

When a capacitor is connected to an inductor as illustrated in Fig. 4.4, the combination is an **LC** circuit. If the capacitor is initially charged and the switch is then closed, we find that both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values. If the resistance of the circuit is zero, no energy is transformed to internal energy. In the following analysis, we neglect the resistance in the circuit. We also assume an idealized situation in which energy is not radiated away from the circuit.

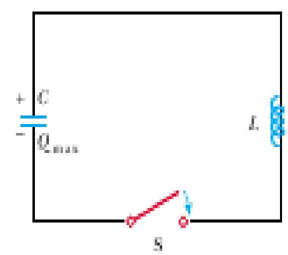


Figure 4.4

When the capacitor is fully charged, the energy **U** in the circuit is stored in the electric field of capacitor and is equal to $\boldsymbol{Q_{max}^2/2C}$. At this time, current in the circuit is zero, and therefore no energy is stored in the inductor. After the switch is closed, the rate at which charges leave enter the capacitor plates (which is also the rate at which the charge on the capacitor changes) is equal to the current in the circuit. As the capacitor begins to discharge after the switch is closed, the energy stored in its electric field decreases. The discharge of the capacitor represents a current in the circuit, and hence some energy is now stored in the magnetic field of the inductor. Thus, energy is transferred from the electric field of the capacitor to the magnetic field of the inductor. When the capacitor is fully discharged, it stores no energy. At this time, the current reaches its maximum value, and all of the energy is stored in the inductor. The current continues in the same direction, decreasing in magnitude, with the capacitor eventually becoming fully charged again but with the polarity of its plates now opposite the initial polarity. This is followed by another discharge until the circuit returns to its original state of maximum charge $\mathbf{Q_{max}}$ and the plate polarity shown in Fig.4.4. The energy continues to oscillate between inductor and capacitor.

Let us consider some arbitrary time t after the switch is closed, so that the capacitor has a charge $\mathbf{Q < Q_{max}}$ and the current is $\mathbf{I < I_{max}}$. At this time, both circuit elements store energy, but the sum of the two energies must equal the total initial energy **U** stored in the fully charged capacitor at $\mathbf{t = 0}$:

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2} L\, I^2 \qquad ----1$$

Because we have assumed the circuit resistance to be zero and we ignore electromagnetic radiation, no energy is transformed to internal energy and none is transferred out of the system of the circuit. Therefore, the total energy of the system must remain constant in time. This means that ***dU/dt = 0***. Therefore, by differentiating Equation1 with respect to time while noting that **Q** and **I** vary with time, we obtain:

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2} L\, I^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} \qquad ----2$$

We can reduce this to a differential equation in one variable by remembering that the current in the circuit is equal to the rate at which the charge on the capacitor changes: $I = dQ/dt$. From this, it follows that $dI/dt = d^2Q/dt^2$. Substitution of these relation- ships into Eq.2 gives:

$$\frac{Q}{C} + L\,\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC} Q \qquad ---3$$

We can solve for **Q** by noting that this expression is of the same form as the analogous Eq. $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ and $\omega^2 = \frac{k}{m}$ for a block-spring system:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

Where **k** is the spring constant, **m** is the mass of the block, and $\boldsymbol{\omega = \sqrt{k/m}}$ the solution of this equation has the general form $\boldsymbol{x(t) = A cos(\omega t + \phi)}$

Where $\boldsymbol{\omega}$ is the angular frequency of the simple harmonic motion, A is the amplitude of motion (the maximum value of **x**), and is the phase constant; the values of A and ϕ depend on the initial conditions. Because Eq.3 is of the same form as the differential equation of the simple harmonic oscillator, we see that it has the solution;

$$\boldsymbol{Q = Q_{max} cos(\omega t + \phi)} \qquad -------4$$

Where $\mathbf{Q}_{max}$ is the maximum charge of the capacitor and the angular frequency $\boldsymbol{\omega}$ is:

$$\boldsymbol{\omega = \frac{1}{\sqrt{LC}}}$$

This is the natural frequency of oscillation of the **LC** circuit. Because **Q** varies sinusoidally with time, the current in the circuit also varies sinusoidally. We can easily show this by differentiating Eq.4 with respect to time:

$$\boldsymbol{I = \frac{dQ}{dt} = -\omega\, Q_{max} sin(\omega t + \phi)} \qquad ----5$$

To determine the value of the phase angle ϕ, we examine the initial conditions, which in our situation require that at t = 0, I =0 and Q = Q $_{\mathbf{max}}$. Setting I = 0 at t = 0 in Eq.5, we have:

$\boldsymbol{0 = -\omega\, Q_{max} sin\, \phi}$ substitute in eq.4 with the condition that $\mathbf{Q = Q_{max}}$ at **t = 0** we get Q and $\boldsymbol{I}$

$$\boldsymbol{Q = Q_{max}\, cos\omega t} \qquad ---------\ 6$$

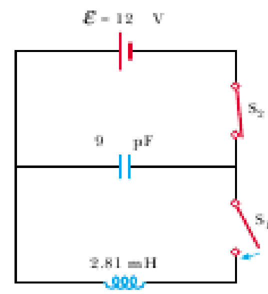
$$\boldsymbol{I = -\omega\, Q_{max} sin\omega t = -I_{max}\, sin\omega t} \qquad --------7$$

Substitute eq.6 and eq.7 in eq.1 we get

$$\boldsymbol{U = U_C + U_L = \frac{Q_{max}^2}{2C} cos^2\omega t + \frac{1}{2} L\, I_{max}^2 sin^2\omega t}$$

Example: The Figure show, the capacitor is initially charged when switch $\mathbf{S_1}$ is open and $\mathbf{S_2}$ is closed. Switch $\mathbf{S_2}$ is then opened, removing the battery from the circuit, and the capacitor remains charged. Switch S_1 is then closed, so that the capacitor is connected directly across the inductor. **(A)** Find the frequency of oscillation of the circuit. **(B)** What are the maximum values of charge on the capacitor and current in the circuit? **(C)** Determine the charge and current as functions of time.

Solution:



$$f=\frac{\omega}{2\pi}=\frac{1}{2\pi\sqrt{LC}}=\frac{1}{2\pi\sqrt{(2.8\times10^{-3}H)(9\times10^{-12}F)}}$$

$$f=1\times10^{-6}Hz$$

$$Q_{max}=C\Delta V=(9\times10^{-12}F)(12V)$$

$$Q_{max}=1.8\times\times10^{-10}C$$

$$I_{max}=\omega Q_{max}=2\pi f Q_{max}$$

$$I_{max}=(2\pi\times10^{6}sec^{-1})(1.8\times10^{-10}C)=6.79\times10^{-4}Amp$$

$$Q=Q_{max}cos\omega t$$

$$Q=(1.8\times\times10^{-10}C)\,cos\{(2\pi\times10^{6}sec^{-1})\times t\}$$

$$I=-I_{max}sin\,\omega t$$

$$I=(6.79\times10^{-4}Amp)\,sin(2\pi\times10^{6}sec^{-1})\times t$$

4.6. The RLC Circuit

Let us assume that the resistance of the resistor represents all of the resistance in the circuit, as shown in Fig.4.5. Now imagine that switch $\mathbf{S_1}$ is closed and $\mathbf{S_2}$ is open, so that the capacitor has an initial charge Q max. Next, $\mathbf{S_1}$ is opened and $\mathbf{S_2}$ is closed. Once $\mathbf{S_2}$ is closed and a current is established, the total energy stored in the capacitor and inductor at any time is given by Eq. $\boldsymbol{U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2} L\, I^2}$. However, this total energy is no longer constant, as it was in the **LC** circuit, because the resistor causes transformation to internal energy. Because the rate of energy transformation to internal energy within a resistor is $\mathbf{I^2R}$, we have:

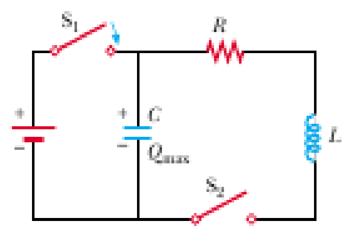


Figure 4.5

$$\boldsymbol{\frac{dU}{dt} = -I^2R}$$

Where the negative sign signifies that the energy **U** of the circuit decreasing in time. Substituting this result into Eq.

$$\boldsymbol{\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2} L\, I^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0}$$ gives:

$$\boldsymbol{\frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = -I^2R \qquad -----1}$$

Substituted $\boldsymbol{\frac{dQ}{dt}} = I$ in eq.1 we get

$$\boldsymbol{LI\frac{d^2Q}{dt^2} + \frac{Q}{C}I + I^2R = 0 \qquad \div I}$$

$$\boldsymbol{L\frac{d^2Q}{dt^2} + \frac{Q}{C} + IR = 0}$$