Induced Electromotive Force

3.1. Faraday's Law of Induction

The **emf** induced in a circuit is directly proportional to the time rate of change of the magnetic flux through the circuit.

This statement, known as Faraday's law of induction, can be written

$$\varepsilon = -\frac{d\phi_B}{dt}$$

Where $\phi_B = \int B dA$ Is the magnetic flux through the circuit. If the circuit is a coil consisting of **N** loops all of the same area and if ϕ_B Is the magnetic flux through one loop, an **emf** is induced in every loop. The loops are in the series, so their **emf** add; thus, the total induced **emf** in the coil is given by the expression:

$$\varepsilon = -N\frac{d\phi_B}{dt}$$

Suppose that a loop enclosing an area **A** lies in a uniform magnetic field **B**, as in Figure 3.1. The magnetic flux through the loop is equal to **BA cosθ**; hence, the induced **emf** can be expressed as:

$$\varepsilon = -N\frac{d}{dt}(BA\cos\theta)$$

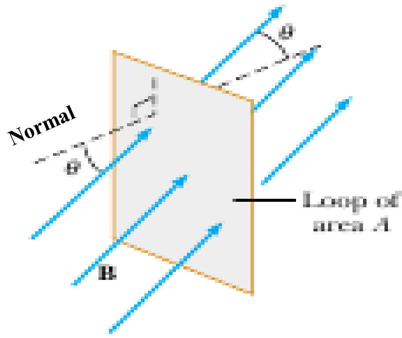


Figure 3.1

From this expression, we see that an **emf** can be induced in the circuit in several ways:

1.The magnitude of **B** can change with time.

2.The area enclosed by the loop can change with time.

3. The angle **θ** between **B** and the normal to the loop can change with time.

4. Any combination of the above can occur.

Example: A coil of **15 turns** and radius **10cm** surrounds a long solenoid of radius **2cm** and **1x10^3 turns/meter** (show in the Figure) The current in the solenoid changes as **I =(5.00 A)sin(120t)**. Find the induced **emf** in the **15-turn** coil as a function of time.

Solution:

$$\Phi_B = (\mu_o nI)A_{soenoid}$$

$$\Phi_B = -N\frac{d\Phi_B}{dt} = -N\mu_o n(\pi r^2)\frac{dI}{dt}$$

$$\Phi_B = -15\left(4\pi \times 10^{-7}T.\frac{m}{A}\right)(1 \times 10^3 m^{-1})\pi(0.02m)^2\left(\frac{600A}{s}\right)cos120t$$

$$\varepsilon = -14.2cos(120t)mV$$

3.2. Induced emf and Electric Fields

We have seen that a changing magnetic flux induces an **emf** and a current in a conducting loop. In our study of electricity, we related a current to an electric field that applies electric forces on charged particles. In the same way, we can relate an induced current in a conducting loop to an electric field by claiming that an electric field is created in the conductor as a result of the changing magnetic flux.

This induced electric field is **non conservative**, unlike the electrostatic field produced by stationary charges. We can illustrate this point by considering a conducting loop of radius r situated in a uniform magnetic field that is perpendicular to the plane of the loop, as in Figure 3.1. If the magnetic field changes with time, then, according to Faraday's law, an **emf** $\mathcal{E} = -d\phi_B/dt$ is induced in the loop. The induction of a current in the loop implies the presence of an induced electric field **E**, which must be tangent to the loop because this is the direction in which the charges in the wire move in response to the electric force. The work done by the electric field in moving a test charge **q** once around the loop is equal to **q$\mathcal{E}$**. Because the electric force acting on the charge is **qE**, the work done by the electric field in moving the charge once around the loop is **qE(2πr),** where **2πr** is the circumference of the loop. These two expressions for the work done must be equal; therefore, we see that

$$\mathbf{q}\mathcal{E} = \mathbf{qE}(\mathbf{2\pi r})$$

$$\mathbf{E} = \frac{\mathcal{E}}{\boldsymbol{2\pi r}}$$

$$\boldsymbol{\phi = BA = \pi r^2 B}$$

$$\mathbf{E} = -\frac{\mathbf{1}}{\boldsymbol{2\pi r}}\frac{\boldsymbol{d\phi_B}}{\boldsymbol{dt}} = -\frac{\boldsymbol{r}}{\mathbf{2}}\frac{\boldsymbol{dB}}{\boldsymbol{dt}}$$

The **emf** for any closed path can be expressed as the line integral of **E.ds** over that path: $\mathcal{E} = \oint \boldsymbol{E.ds}$. In more general cases, **E** may not be constant, and the path may not be a circle. Hence, Faraday's law of induction, $\mathcal{E} = -\boldsymbol{d\phi_B/dt}$, can be written in the general form

$$\oint \boldsymbol{E.ds} = -\frac{\boldsymbol{d\phi_B}}{\boldsymbol{dt}} \qquad --\mathbf{1}$$

The induced electric field **E** in Equation **1** is a **nonconservative field** that is generated by a changing magnetic field. The field **E** that satisfies Equation**1** cannot possibly be an electrostatic field because if the field were electrostatic, and hence conservative, the line integral of **E. ds** over a closed loop would be zero.

Example: A long solenoid with **1000 turns** per meter and radius **2cm** carries an oscillating current given by **I= (5A) sin(100πt)**. What is the electric field induced at a radius **r =1cm** from the axis of the solenoid? What is the direction of this electric field when the current is increasing counterclockwise in the coil?

Solution: $\oint E.dt = |\frac{d\Phi_B}{dt}|$

$$2\pi r E = (\pi r^2)\frac{dB}{dt} \quad \Rightarrow E = \left(\frac{9.87mV}{m}\right)(cos100\pi t)$$

The **E** field is always opposite to increasing **B**. ⇒ **clockwise**

3.3.Lenz's Law

The induced current produces magnetic fields which tend to oppose the change in magnetic flux that induces such currents.

To illustrate how Lenz's law works, let's consider a conducting loop placed in a magnetic field. We follow the procedure below:

1. Define a positive direction for the area vector $\vec{A}$.
2. Assuming that $\vec{B}$ is uniform, take the dot product of $\vec{B}$ and $\vec{A}$. This allows for the determination of the sign of the magnetic flux Φ_B. .
3. Obtain the rate of flux change $\frac{d\Phi_B}{dt}$ by differentiation. There are three possibilities:

$$\frac{d\Phi_B}{dt}: \begin{cases} > 0 \;\Rightarrow\; \text{induced emf } \varepsilon < 0 \\ < 0 \;\Rightarrow\; \text{induced emf } \varepsilon > 0 \\ = 0 \;\Rightarrow\; \text{induced emf } \varepsilon = 0 \end{cases}$$

4. Determine the direction of the induced current using the right-hand rule. With your thumb pointing in the direction of $\vec{A}$, curl the fingers around the closed loop. The induced current flows in the same direction as the way your fingers curl if **ε >0**, and the opposite direction if **ε < 0**, as shown in Figure.

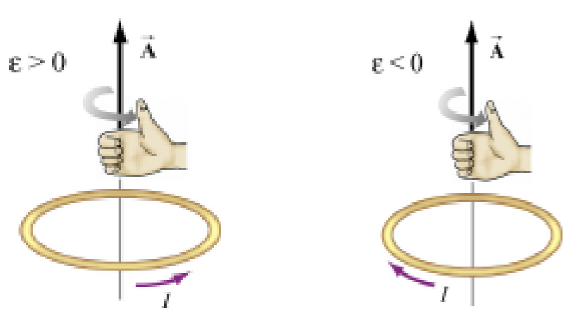


Figure 3.2. Determination of the direction of induced current by the right-hand rule

3.3.1. Motional EMF

Consider a conducting bar of length l moving through a uniform magnetic field which points into the page, as shown in Figure 3.3. Particles with charge q > 0 inside experience a magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ which tends to push them upward, leaving negative charges on the lower end.

The separation of charge gives rise to an electric field $\vec{E}$ inside the bar, which in turn a downward electric force $\vec{F} = q\vec{E}$. equilibrium where the two forces cancel,

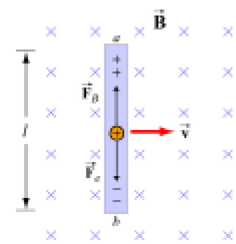


Figure 3.3. A conducting bar moving through a uniform magnetic field

We have $qvB = qE$, or $E = vB$. the two ends of the conductor, there exists a potential difference given by:

$$V_{ab} = V_a - V_b = \varepsilon = E\ell = vB\ell$$

Since ε arises from the motion of the conductor, this potential difference is called the motional emf. In general, motional emf around a closed conducting loop can be written as:

$$\varepsilon = \oint (\vec{v} \times \vec{B}) . d\vec{s}$$

Where $d\vec{s}$ is a differential length element.

Now suppose the conducting bar moves through a region of uniform magnetic field $\vec{B} = B\hat{k}$ (pointing into the page) by sliding along two frictionless conducting rails that are at a distance ℓ apart and connected together by a resistor with resistance **R**, as shown in Figure 3.4.

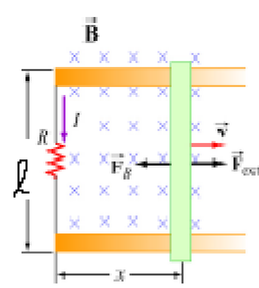


Figure 3.4. A conducting bar sliding along two conducting rails

Let an external force be applied so that conductor moves to the right with a constant velocity $\vec{v} = v\hat{i}$. The magnetic flux through the loop formed by the bar and the rails is given by:

$$\Phi_B = BA = B\ell x$$

Thus, according to Faraday's law, the induced **emf** is:

$$\boldsymbol{\varepsilon} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B\ell x) = -BI\frac{dx}{dt} = -B\ell v$$

Where $dx/dt = v$ is simply the speed of the bar. The corresponding induced current is:

$$I = \frac{|\varepsilon|}{R} = \frac{B\ell v}{R}$$

Example: A metal ring is placed near a solenoid, as shown in Figure 3.2 Find the direction of the induced current in the ring. **(A)** at the instant the switch in the circuit containing the solenoid is thrown closed, **(B)** after the switch has been closed for several seconds, and **(C)** at the instant the switch is thrown open.

Solution: **(A)** At the instant the switch is thrown closed, the situation changes from one in which no magnetic flux exists in the ring to one in which flux exists and the magnetic field is to the left as shown in Figure **3.2b**. To counteract this change in the flux, the current induced in the ring must set up a magnetic field directed from left to right in Figure **3.2b**. This requires a current directed as shown. **(B)** After the switch has been closed for several seconds, no change in the magnetic flux through the loop occurs; hence, the induced current in the ring is zero. **(C)** Opening the switch changes the situation from one in which magnetic flux exists in the ring to one in which there is no magnetic flux. The direction of the induced current is as shown in Figure **3.2c** because current in this direction produces a magnetic field that is directed right to left and so counteracts the decrease in the flux produced by the solenoid.

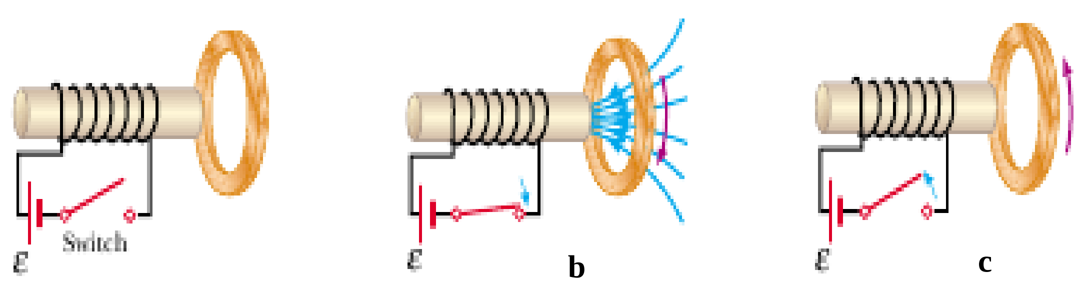
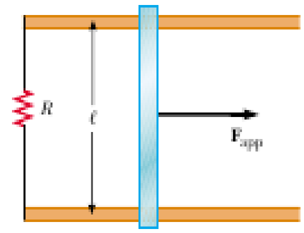


Figure 3.2

Example: Figure shows a top view of a bar that can slide without friction. The resistor is **6Ω** and a **2.5T** magnetic field is directed perpendicularly downward, into the paper. Let **ℓ=1.2m**. (a) Calculate the applied force required to move the bar to the right at a constant speed of **2m/s**. (b) At what rate is energy delivered to the resistor?



Solution:

a. $|F_B| = I|\ell \times B|$

$$When\ I = \frac{\varepsilon}{R} \quad and\ \varepsilon = B\ell v$$

$$We\ get\ F_B = \frac{B\ell v}{R}(\ell v) = \frac{B^2\ell^2 v}{R}$$

$$F_B = \frac{(2.5)^2(1.2)^2(2)}{6} = 3N \quad to\ the\ right$$

b. $P = I^2R = \frac{B^2\ell^2v^2}{R} = 6W$