1. Physics Laws for The Neutral Atmosphere

Characterizing the atmosphere by the radio waves are propagated leads to a subdivision troposphere and ionosphere. The ionosphere, the upper part of the atmosphere, is the dispersive medium (the propagation delay is frequency dependent) whereas the troposphere is non-dispersive. The troposphere is also referred to as neutral atmosphere.
In this chapter, the following physical laws and equations are given the gas equation (equation of state), hydrostatic equation, and Snell's law

1-1 Equation of state

Laboratory experiments show that the pressure, volume, and temperature of any material can be related by an equation of state over a wide range of conditions.
All gases are found to follow approximately the same equation of state, which is referred to as the ideal gas equation. For most purposes, we may assume that atmospheric gases, whether considered individually or as a mixture, obey the ideal gas equation exactly. This section considers various forms of the ideal gas equation and its application to dry and moist air.

There are three laws for gases are given:

- Charles' constant pressure law: "At constant pressure for a rise in temperature of 1degree Celsius, all gases expand by a constant amount, equal to 1/ 273 of their volume at 0 degree Celsius".
- Charles' constant volume law: "If the volume is kept constant, all gases undergo an increase in pressure equal to 1/ 273 of their pressure at 0 degree Celsius".
- Boyle's law: "At constant temperature the product of pressure and volume is constant".

Based on these laws, the gas equation of state is formulated for perfect gases:

$$PV = mRT \qquad (1)$$

where P, V, m, and T are the pressure (Pa), volume (m^3), mass (kg), and absolute temperature (in kelvin, K, where K = °C + 273.15) of the gas, respectively, and R is a constant (called the gas constant) for 1 kg of a gas. The value of R depends on the particular gas under consideration. Because $m/V = \rho$, where ρ is the density of the gas, the ideal gas equation may also be written in the form

$$P = \rho RT \qquad (2)$$

For a unit mass (1 kg) of gas m=1 and we may write (2) as:

$$P\alpha = RT \qquad (3)$$

where $\alpha = 1/\rho$ is the *specific volume* of the gas, i.e., the volume occupied by 1 kg of the gas at pressure p and temperature T.

If the temperature is constant (1) reduces to Boyle's law, which states if the temperature of a fixed mass of gas is held constant, the volume of the gas is inversely proportional to its pressure. Changes in the physical state of a body that occur at constant temperature are termed isothermal. Also, implicit in (1) are Charles' two laws. The first of these laws states for a fixed mass of gas at constant pressure, the volume of the gas is directly proportional to its absolute temperature. The second of Charles' laws states for a fixed mass of gas held within a fixed volume, the pressure of the gas is proportional to its absolute temperature.

We define now a gram-molecular weight or a mole (abbreviated to mol) of any substance as the molecular weight, M, of the substance expressed in grams. For example, the molecular weight of water is 18.015; therefore, 1 mol of water is 18.015 g of water. The number of moles *n* in mass *m* (in grams) of a substance is given by

$$n = \frac{m}{M} \qquad (4)$$

Because the masses contained in 1 mol of different substances bear the same ratios to each other as the molecular weights of the substances, 1 mol of any substance must contain the same number of molecules as 1 mol of any other substance. Therefore, the number of molecules in 1 mol of any substance is a universal constant, called Avogadro's number, N_A. The value of N_A is 6.022×10^{23} per mole.

According to *Avogadro's hypothesis*, gases containing the same number of molecules occupy the same volumes at the same temperature and pressure. It follows from this hypothesis that provided we take the same number of molecules of any gas, the constant R in (1) will be the same. However, 1 mol of any gas contains the same number of molecules as 1 mol of any other gas. Therefore, the constant R in (1) for 1 mol is the same for all gases; it is called the *universal gas constant* (R*).

The magnitude of R* is $8.3145\,J\ K^{-1}\,mol^{-1}$. The ideal gas equation for 1 mol of any gas can be written as

$$PV = R^*T \qquad (5)$$

and for *n* moles of any gas as

$$PV = nR^*T \qquad (6)$$

The gas constant for one molecule of any gas is also a universal constant, known as Boltzmann's constant, k.

Because the gas constant for N_A molecules is R*, we have

$$k = \frac{R^*}{N_A} \qquad (7)$$

Hence, for a gas containing n_0 (number density) molecules per unit volume, the ideal gas equation is

$$P = n_0 kT \qquad (8)$$

Prove??

If the pressure and specific volume of dry air (i.e., the mixture of gases in air, excluding water vapor) are P_d and α_d, respectively, the ideal gas equation in the form of (3) becomes

$$P_d\alpha_d = R_dT \qquad (9)$$

where R_d is the gas constant for 1 kg of dry air. By analogy with (4), we can define the *apparent molecular weight* M_d of dry air as the total mass (in grams) of the constituent gases in dry air divided by the total number of moles of the constituent gases; that is,

$$M_d = \frac{\sum_i m_i}{\sum_i \frac{m_i}{M_i}} \qquad (10)$$

where m_i and M_i represent the mass (in grams) and molecular weight, respectively, of the ith constituent in the mixture. The apparent molecular weight of dry air is 28.97. Because R^* is the gas constant for 1 mol of any substance, or for M_d (= 28.97) grams of dry air, the gas constant for 1 g of dry air is R^*/M_d, and for 1 kg of dry air it is

$$R_d = 1000\frac{R^*}{M_d} = 1000\frac{8.3145}{28.97} = 287.0\,JK^{-1}Kg^{-1} \qquad (11)$$

The ideal gas equation may be applied to the individual gaseous components of air. For example, for water vapor (3) becomes

$$e\alpha_v = R_v T \qquad (12)$$

where e and α_v are, respectively, the pressure and specific volume of water vapor and R_v is the gas constant for 1 kg of water vapor. Because the molecular weight of water is $M_w (= 18.016)$ and the gas constant for M_w grams of water vapor is $R*$, we have

$$R_v = 1000 \frac{R^*}{M_w} = 1000 \frac{8.3145}{18.016} = 461.51\, JK^{-1} Kg^{-1} \qquad (13)$$

From (11) and (13),

$$\frac{R_d}{R_v} = \frac{M_w}{M_d} = \varepsilon = 0.622 \qquad (14)$$

Because air is a mixture of gases, it obeys Dalton's law of partial pressures, which states the total pressure exerted by a mixture of gases that do not interact chemically is equal to the sum of the partial pressures of the gases. The partial pressure of a gas is the pressure it would exert at the same temperature as the mixture if it alone occupied all of the volume that the mixture occupies.

Exercise 1: If at 0 °C the density of dry air alone is $1.275\ kg\ m^{-3}$ and the density of water vapor alone is $4.770 \times 10^{-3} kg\ m^{-3}$, what is the total pressure exerted by a mixture of the dry air and water vapor at 0 °C?

Solution: From Dalton's law of partial pressures, the total pressure exerted by the mixture of dry air and water vapor is equal to the sum of their partial pressures. The partial pressure exerted by the dry air is, from (9),

$$P_d = \frac{1}{\alpha_d} R_d T = \rho_d R_d T$$

where ρ_d is the density of the dry air ($1.275\ kg\ m^{-3}\ at\ 273\ K$), R_d is the gas constant for 1 kg of dry air ($287.0\ J\ K^{-1}\ kg^{-1}$), and T is 273.2 K. Therefore,

$$P_d = 9.997 \times 10^4 Pa = 999.7\ hPa$$

Similarly, the partial pressure exerted by the water vapor is, from (12),

$$e = \frac{1}{\alpha_v} R_v T = \rho_v R_v T$$

where ρ_v is the density of the water vapor ($4.770 \times 10^{-3}\ kg\ m^{-3}\ at\ 273\ K$), R_v is the gas constant for 1 kg of water vapor ($461.5\ J\ K^{-1}\ kg^{-1}$), and T is 273.2 K. Therefore,

$$e = 601.4 Pa = 6.014\ hPa$$

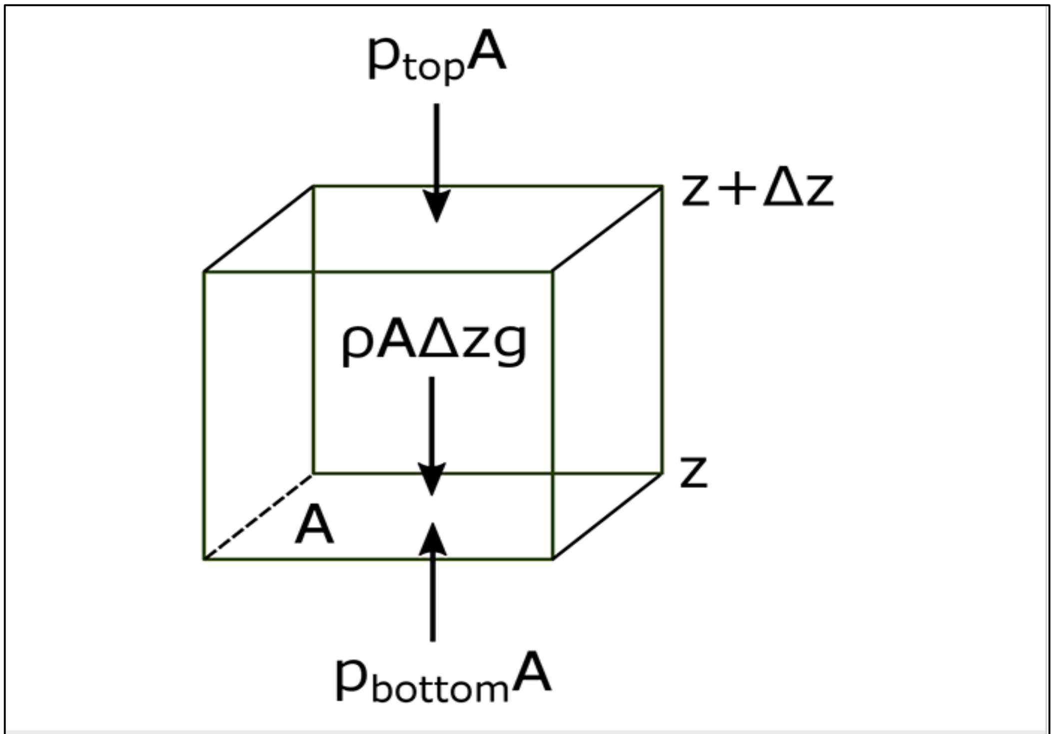
Hence, the total pressure exerted by the mixture of dry air and water vapor is (999.7+6.014) hPa or 1006 hPa.

Exercise 2: Determine the apparent molecular weight of the Venusian atmosphere, assuming that it consists of 95% of CO2 and 5% N2 by volume. What is the gas constant for 1 kg of such an atmosphere? (Atomic weights of C, O, and N are 12, 16, and 14, respectively.)

Solve??

1-2 Hydrostatic Equilibrium

The atmosphere's basic pressure structure is determined by the hydrostatic balance of forces. To a good approximation, every air parcel is acted on by three forces that are in balance, leading to no net force. Since they are in balance for any air parcel, the air can be assumed to be static or moving at a constant velocity



Air parcel at rest with three forces in balance

There are 3 forces that determine hydrostatic balance:

1. One force is downwards (negative) onto the top of the cuboid from the pressure, p, of the fluid above it. It is, from the definition of pressure
$$F_{top} = -P_{top}.A \quad (1)$$
2. Similarly, the force on the volume element from the pressure of the fluid below pushing upwards (positive) is:
$$F_{bottom} = P_{bottom}.A \quad (2)$$

3. Finally, the weight of the volume element causes a force downwards. If the density is ρ, the volume is V, which is simply the horizontal area A times the vertical height, Δz, and g the standard gravity, then:

$$F_{gravity} = -\rho V g = -\rho g A \Delta z \qquad (3)$$

$$F_{gravity} = -mg$$
$$\therefore m = \rho V$$
$$\therefore V = A\Delta z$$
so $F_{gravity} = -\rho g A \Delta z$

By balancing these forces, the **total force on the fluid** is:

$$\sum F = F_{bottom} + F_{top} + F_{gravity} = P_{bottom}.A - P_{top}.A - \rho g A \Delta z \qquad (4)$$

This sum equals zero if the air's velocity is constant or zero. Dividing by A,

$$0 = P_{bottom} - P_{top} - \rho g \Delta z \qquad (5)$$

or:

$$P_{top} - P_{bottom} = -\rho g \Delta z \qquad (6)$$

$P_{top} - P_{bottom}$ is a change in pressure, and Δz is the height of the volume element – a change in the distance above the ground. By saying these changes are infinitesimally small, the equation can be written in differential form, where dp is top pressure minus bottom pressure just as dz is top altitude minus bottom altitude.

$$dp = -\rho g dz \qquad (7)$$

The result is the equation:

$$\frac{dp}{dz} = -\rho g \qquad (8)$$

This equation is called the **Hydrostatic Equation**.

Using the Ideal Gas Law, we can replace ρ and get the equation for dry air:

$$\frac{dp}{dz} = -g\frac{P}{R_d T} \qquad (9)$$

or

$$\frac{dp}{p} = -\frac{g}{R_d T}\,dz = -\frac{Mg}{R^* T}\,dz \qquad (10)$$

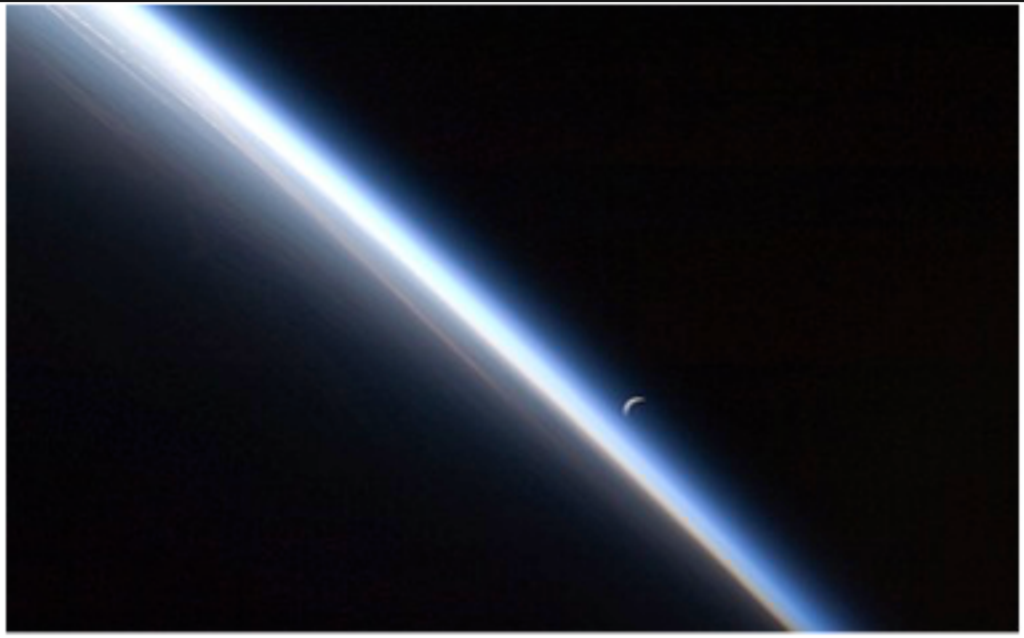
Prove??

We could integrate both sides to get the altitude dependence of p, but we can only do that if T is constant with height. It is not, but it does not vary by more than about ±20%. So, doing the integral,

$$p = p_0 e^{-\frac{z}{H}} \quad where\ p_0 = surface\ pressure\ and\ \ H = \frac{R^*T}{M_{air}\, g}$$

H is called a scale height because when $z = H$, we have e^{-1}. If we use an average T of 250 K, with $M_{air} = 0.029$ kg mole^{-1}, then H = 7.2 km. The pressure at this height is about 360 hPa, close to the 300 mb surface that you have seen on the weather maps. of course, the forces are not always in hydrostatic balance and the pressure depends on temperature, thus the pressure changes from one location to another on a constant height surface.

From the hydrostatic equation, the atmospheric pressure falls off exponentially with height, which means that about every 7 km, the atmospheric pressure is about 1/3 less. At 40 km, the pressure is only a few tenths of a percent of the surface pressure. Similarly, the concentration of molecules is only a few tenths of a percent, and since molecules scatter sunlight, you can see in the picture below that the scattering is much greater near Earth's surface than it is high in the atmosphere.

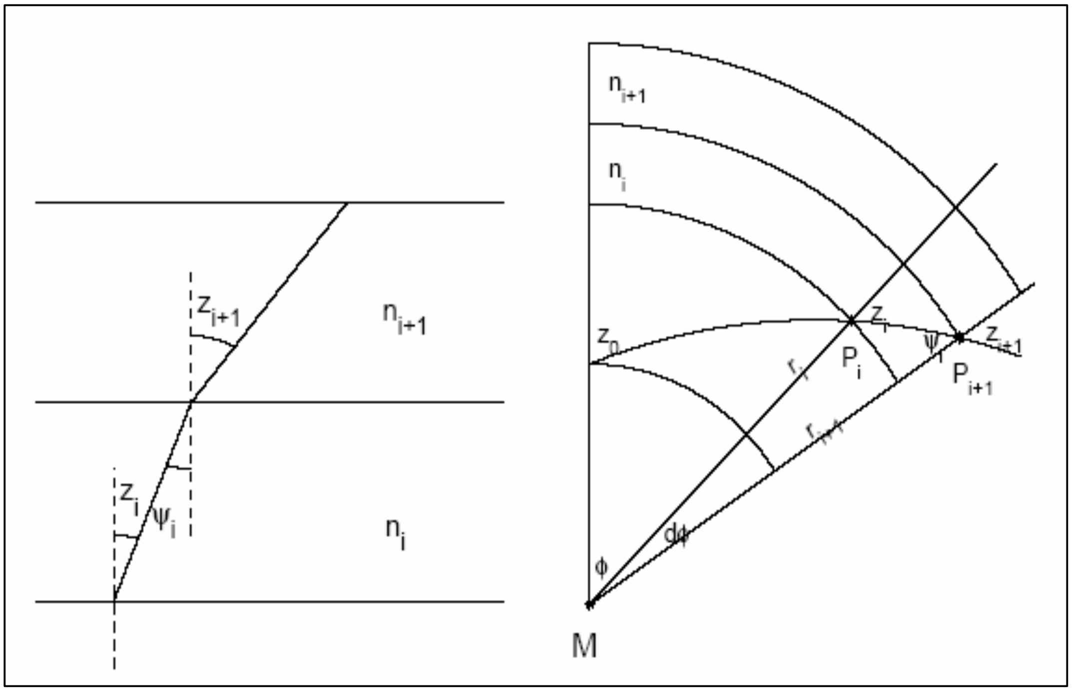


Scattered light near Earth's surface.

1-3 Snell's law

A radio signal passing through the Earth's atmosphere suffers a change in direction owing to refraction.

If we consider the neutral atmosphere to be horizontally stratified and neglect the ionospheric refraction, the total bending can be found by repeatly applying Snell's law foe each layer (Smart,1936).



Snell's law states:

$$n_{i+1} \sin z_{i+1} = n_i \sin z_i = n_i \sin \psi_i \qquad (1)$$

where z_i and z_{i+1} are the zenith angle of the arriving radio signal in the layer i and are the corresponding refractive indexes.

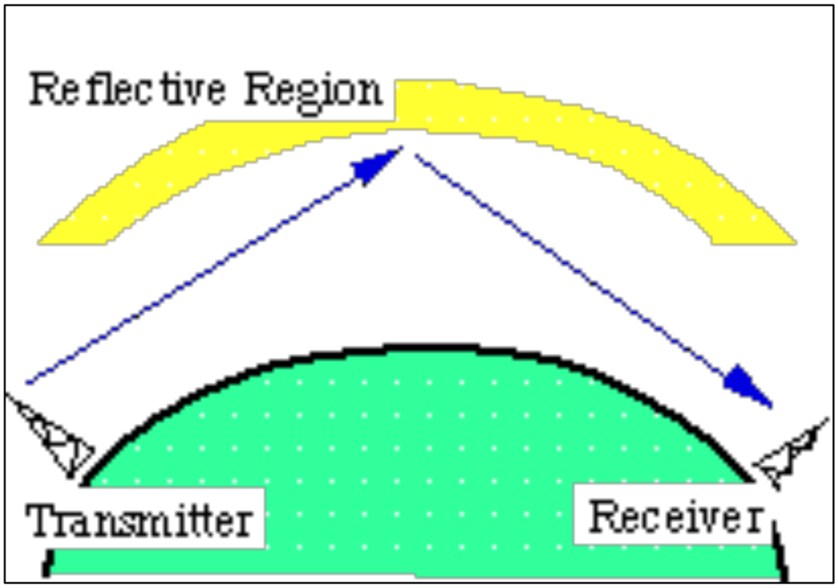
$$n_0 \sin z_0 = n_m \sin z_m \qquad (2)$$

where the index n_0 denotes the lowest layer and the index n_m denotes the highest layer, when the refractive index reduces to 1. The formula (2) will be:

$$n_0 \sin z_0 = \sin z_m \qquad (3)$$

This formula (3) holds for any refractivity profile.

For a spherical Earth, we may formulate Snell's law in spherical coordinates.



Application of sine rule in the triangle MP_iP_{i+1} for this figure gives:

$$\frac{r_i}{\sin\psi_i} = \frac{r_{i+1}}{\sin(\pi - z_i)} = \frac{r_{i+1}}{\sin z_i} \qquad (4)$$

where r_i and r_{i+1} are the distance MP_i *and* MP_{i+1} *with* M two center of mass of the Earth. Combining Eq. (1) and (4) gives Snell's law in spherical coordinates:

$$n_{i+1}\, r_{i+1} \sin z_{i+1} = n_i\, r_i \sin z_i = n_0\, r_0 \sin z_0 \qquad (5)$$

or simply

$$n_0 \sin z_0 = \sin z_m \qquad (6)$$

1-4 Virtual Temperature

Moist air has a smaller apparent molecular weight than dry air. Therefore, it follows from (eq. 13 in page 4)

$$R_v = 1000\frac{R^*}{M_w} = 1000\frac{8.3145}{18.016} = 461.51\,JK^{-1}Kg^{-1} \qquad (13)$$

that the gas constant for 1 kg of moist air is larger than that for 1 kg of dry air. However, rather than use a gas constant for moist air, the exact value of which would depend on the amount of water vapor in the air (which varies considerably), it is convenient to retain the gas constant for dry air and use a fictitious temperature (called the virtual temperature) in the ideal gas equation. We can derive an expression for the virtual temperature in the following way.

Consider a volume V of moist air at temperature T and total pressure p that contains mass md of dry air and mass mv of water vapor. The density of the moist air is given by

$$\rho = \frac{m_d + m_v}{V} = \rho_d' + \rho_v'$$

where ρ_d' is the density that the same mass of dry air would have if it alone occupied all of the volume V and ρ_v' is the density that the same mass of water vapor would have if it alone occupied all of the volume V. We may call these ***partial densities***. Because $\rho = \rho_d' + \rho_v'$, it might appear that the density of moist air is greater than that of dry air. However, this is not the case because the partial density ρ_v' is less than the true density of dry air. Applying the ideal gas equation in the form of (eq. 2 in page 2, $P = \rho RT$) to the water vapor and dry air in turn, we have

$$e = \rho_v' R_v T$$

and

$$P_d' = \rho_d' R_d T$$

where *e* and P_d' are the partial pressures exerted by the water vapor and the dry air, respectively. Also, from Dalton's law of partial pressures,

$$P = P_d' + e$$

Combining the last four equations

$$\rho = \frac{p-e}{R_d T} + \frac{e}{R_v T}$$

or

$$\rho = \frac{P}{R_d T}\left[1 - \frac{e}{P}(1-\varepsilon)\right]$$

Prove??

where ε is defined by (eq.14 in page 4, $\frac{R_d}{R_v} = \frac{M_w}{M_d} = \varepsilon = 0.622$). The last equation may be written as

$$P = \rho R_d T_v$$

where

$$T_v = \frac{T}{1 - \frac{e}{P}(1-\varepsilon)}$$

T_v is called the ***virtual temperature***.

The great advantage of introducing virtual temperature is that the total pressure and total density of the mixture are related by the ideal gas equation with the gas constant the same as that for dry air, R_d. The virtual temperature is the temperature that dry air must have in order to to have the same density as the moist air at the same pressure. Note that the virtual temperature is always greater than the actual temperature:

$$T_v \geq T$$

Typically, the virtual temperature exceeds the actual temperature by only a few degrees.