Examples:

Ex. (1): Express the vector field $\mathbf{A} = yz\, i - y\, j + xz^2\, k$ in cylindrical polar coordinates, and hence calculate its divergence?

Solution: $x = \rho \cos\varphi\,, \quad y = \rho\, \sin\varphi\,, \quad z = z$

$$\mathbf{r} = \rho \cos\varphi\ i + \rho\ \sin\varphi\ j + z\, k$$

$$\hat{e}_\rho = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left|\frac{\partial \mathbf{r}}{\partial \rho}\right|} = \cos\varphi\ i + \sin\varphi\ j$$

$$\hat{e}_\varphi = \frac{\frac{\partial \mathbf{r}}{\partial \varphi}}{\left|\frac{\partial \mathbf{r}}{\partial \varphi}\right|} = -\sin\varphi\ i + \cos\varphi\ j$$

$$\hat{e}_z = \frac{\frac{\partial \mathbf{r}}{\partial z}}{\left|\frac{\partial \mathbf{r}}{\partial z}\right|} = k$$

$$\therefore i = \cos\varphi\ \hat{e}_\rho - \sin\varphi\ \hat{e}_\varphi\,, \qquad j = \sin\varphi\ \hat{e}_\rho + \cos\varphi\ \hat{e}_\varphi\,, \qquad k = \hat{e}_z$$

$$\mathbf{A} = yz\, i - y\, j + xz^2 k = \rho \sin\varphi\, z\left(\cos\varphi\, \hat{e}_\rho - \sin\varphi\, \hat{e}_\varphi\right) - \rho \sin\varphi\left(\sin\varphi\, \hat{e}_\rho + \cos\varphi\, \hat{e}_\varphi\right) + \rho\cos\varphi\, z^2 \hat{e}_z$$

$$\mathbf{A} = (z\rho \sin\varphi \cos\varphi - \rho \sin^2\varphi)\hat{e}_\rho - (z\rho \sin^2\varphi + \rho \sin\varphi\cos\varphi)\hat{e}_\varphi + z^2\rho\cos\varphi\, \hat{e}_z$$

For cylindrical Coordinate

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\ (\rho A_1) + \frac{\partial}{\partial \varphi}\ (A_2) + \frac{\partial}{\partial z}\ (\rho A_3)\right]$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\ \left(\rho(z\rho \sin\varphi\cos\varphi - \rho\sin^2\varphi)\right) + \frac{\partial}{\partial \varphi}\ (z\rho\sin^2\varphi + \rho\sin\varphi\cos\varphi) + \frac{\partial}{\partial z}\ \left(\rho(z^2\rho\cos\varphi)\right)\right]$$

$$\nabla \cdot \mathbf{A} = 2z\sin\varphi\cos\varphi - 2\sin^2\varphi - 2z\sin\varphi\cos\varphi - \cos^2\varphi + \sin^2\varphi + 2z\rho\cos\varphi$$

$$\therefore \nabla \cdot \mathbf{A} = 2z\rho\cos\varphi - 1$$

Ex. (2): Express the vector field $\mathbf{A} = x\, i + 2y\, j + yz\, k$ in spherical polar coordinates?

Solution: $x = r\ \sin\theta \cos\emptyset\ ,\ y = r\ \sin\theta\ \sin\emptyset\ ,\ \ z = r\ \cos\theta$

$$\mathbf{r} = r\ \sin\theta \cos\emptyset\ i + r\ \sin\theta\ \sin\emptyset\ j + r\ \cos\theta\ k$$

$$\hat{e}_r = \frac{\frac{\partial \mathbf{r}}{\partial r}}{\left|\frac{\partial \mathbf{r}}{\partial r}\right|} = \sin\theta \cos\emptyset\ i + \sin\theta\ \sin\emptyset\ j + \cos\theta\ k$$

$$\hat{e}_\theta = \frac{\frac{\partial \mathbf{r}}{\partial \theta}}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|} = \cos\theta \cos\emptyset\ i + \cos\theta\ \sin\emptyset\ j - \sin\theta\ k$$

$$\hat{e}_\emptyset = \frac{\frac{\partial \mathbf{r}}{\partial \emptyset}}{\left|\frac{\partial \mathbf{r}}{\partial \emptyset}\right|} = -\ \sin\emptyset\ i + \cos\emptyset\ j$$

$$\therefore\ i = \sin\theta \cos\emptyset\ \hat{e}_r + \cos\theta \cos\emptyset\ \hat{e}_\theta -\ \sin\emptyset\, \hat{e}_\emptyset$$

$$j = \sin\theta\ \sin\emptyset\ \hat{e}_r + \cos\theta\ \sin\emptyset\ \hat{e}_\theta + \cos\emptyset\ \hat{e}_\emptyset$$

$$k = \cos\theta\, \hat{e}_r - \sin\theta\ \hat{e}_\theta$$

$$\mathbf{A} = x\, i + 2y\, j + yz\, k$$

$$= r\ \sin\theta \cos\emptyset\,(\sin\theta \cos\emptyset\ \hat{e}_r + \cos\theta \cos\emptyset\ \hat{e}_\theta -\ \sin\emptyset\, \hat{e}_\emptyset)$$

$$+2r\ \sin\theta\ \sin\emptyset\,(\sin\theta\ \sin\emptyset\ \hat{e}_r + \cos\theta\ \sin\emptyset\ \hat{e}_\theta + \cos\emptyset\ \hat{e}_\emptyset)$$

$$+(r\ \sin\theta\ \sin\emptyset)(r\ \cos\theta)(\cos\theta\, \hat{e}_r - \sin\theta\ \hat{e}_\theta)$$

$$\mathbf{A} = (r\sin^2\theta\cos^2\emptyset + 2r\sin^2\theta\sin^2\emptyset + r^2 \sin\theta\ \sin\emptyset \cos^2\theta)\hat{e}_r$$

$$+(r \sin\theta \cos\theta \cos^2\emptyset + 2r \sin\theta \cos\theta \sin^2\emptyset - r^2 \sin\emptyset\ \cos\theta \cos^2\theta)\hat{e}_\theta$$

$$+(-r \sin\theta \sin\emptyset\ \cos\emptyset + 2r \sin\theta \sin\emptyset\ \cos\emptyset)\hat{e}_\emptyset$$

$$\mathbf{A} = (r\sin^2\theta(1 + \sin^2\emptyset) + r^2 \sin\theta\ \sin\emptyset \cos^2\theta)\hat{e}_r$$

$$+(r \sin\theta \cos\theta\,(1 + \sin^2\emptyset) - r^2 \sin\emptyset\ \cos\theta \cos^2\theta)\hat{e}_\theta$$

$$+(r \sin\theta \sin\emptyset\ \cos\emptyset)\hat{e}_\emptyset$$

Ex. (3): Express the vector field **A** = $z\, i - 2x\, j + y\, k$ in cylindrical polar coordinates?

Solution: $x = \rho \cos\varphi\,, \quad y = \rho\, \sin\varphi\,, \quad z = z$

$$\mathbf{r} = \rho \cos\varphi\;\, i + \rho\;\sin\varphi\;\, j + z\, k$$

$$\hat{e}_\rho = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left|\frac{\partial \mathbf{r}}{\partial \rho}\right|} = \cos\varphi\;\, i + \;\sin\varphi\;\, j$$

$$\hat{e}_\varphi = \frac{\frac{\partial \mathbf{r}}{\partial \varphi}}{\left|\frac{\partial \mathbf{r}}{\partial \varphi}\right|} = -\sin\varphi\;\, i + \cos\varphi\;\, j$$

$$\hat{e}_z = \frac{\frac{\partial \mathbf{r}}{\partial z}}{\left|\frac{\partial \mathbf{r}}{\partial z}\right|} = k$$

$$\therefore i = \cos\varphi\;\,\hat{e}_\rho - \;\sin\varphi\;\,\hat{e}_\varphi\,, \qquad j = \;\sin\varphi\;\,\hat{e}_\rho + \cos\varphi\;\,\hat{e}_\varphi\,, \qquad k = \hat{e}_z$$

$$\mathbf{A} = z\, i - 2x\, j + y\, k = z\left(\cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi\right)$$

$$-2(\rho\cos\varphi)\left(\;\sin\varphi\;\,\hat{e}_\rho + \cos\varphi\;\,\hat{e}_\varphi\right) + \rho\;\sin\varphi\;\,\hat{e}_z$$

$$\mathbf{A} = (z\cos\varphi - 2\rho\sin\varphi\cos\varphi)\hat{e}_\rho - (z\sin\varphi + 2\rho\cos^2\varphi)\hat{e}_\varphi + \rho\;\sin\varphi\;\,\hat{e}_z$$

Ex. (4): Determine the transformation of cylindrical polar coordinates into Cartesian coordinate?

Solution: $\hat{e}_\rho = \cos\varphi\;\, i + \;\sin\varphi\;\, j$

$$\hat{e}_\varphi = -\sin\varphi\;\, i + \cos\varphi\;\, j\;\;\;,\quad \hat{e}_z = k$$

$$\begin{bmatrix}\hat{e}_\rho\\ \hat{e}_\varphi\\ \hat{e}_z\end{bmatrix} = \begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\;\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}i\\ j\\ k\end{bmatrix}$$

$$\begin{bmatrix}i\\ j\\ k\end{bmatrix} = \begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{e}_\rho\\ \hat{e}_\varphi\\ \hat{e}_z\end{bmatrix}$$

$$\therefore i = \cos\varphi\;\,\hat{e}_\rho - \;\sin\varphi\;\,\hat{e}_\varphi\,, \qquad j = \;\sin\varphi\;\,\hat{e}_\rho + \cos\varphi\;\,\hat{e}_\varphi\,, \qquad k = \hat{e}_z$$

Ex. (5): Determine the conversion of spherical polar coordinates into Cartesian coordinate?

Solution: : $x = r\ \sin\theta\cos\emptyset\ ,\ y = r\ \sin\theta\ \sin\emptyset\ ,\ \ z = r\ \cos\theta$

$\hat{e}_r = \sin\theta\cos\emptyset\ i + \sin\theta\ \sin\emptyset\ j + \cos\theta\ k$

$\hat{e}_\theta = \cos\theta\cos\emptyset\ i + \cos\theta\ \sin\emptyset\ j - \sin\theta\ k$

$\hat{e}_\emptyset = -\ \sin\emptyset\ i + \cos\emptyset\ j$

$$\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\emptyset\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\emptyset & \sin\theta\ \sin\emptyset & \cos\theta\\ \cos\theta\cos\emptyset & \cos\theta\ \sin\emptyset & -\sin\theta\\ -\ \sin\emptyset & \cos\emptyset & 0\end{bmatrix}\begin{bmatrix}i\\ j\\ k\end{bmatrix}$$

$$\begin{bmatrix}i\\ j\\ k\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\emptyset & \cos\theta\cos\emptyset & -\ \sin\emptyset\\ \sin\theta\ \sin\emptyset & \cos\theta\ \sin\emptyset & \cos\emptyset\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\emptyset\end{bmatrix}$$

$\therefore\ i = \sin\theta\cos\emptyset\ \hat{e}_r + \cos\theta\cos\emptyset\ \hat{e}_\theta -\ \sin\emptyset\,\hat{e}_\emptyset$

$j = \sin\theta\ \sin\emptyset\ \hat{e}_r + \cos\theta\ \sin\emptyset\ \hat{e}_\theta + \cos\emptyset\ \hat{e}_\emptyset$

$k = \cos\theta\,\hat{e}_r - \sin\theta\ \hat{e}_\theta$

Ex. (6): Find the relation between of cylindrical and spherical coordinates?

Solution:

$$\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{e}_\rho\\ \hat{e}_\varphi\\ \hat{e}_z\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\emptyset & \cos\theta\cos\emptyset & -\ \sin\emptyset\\ \sin\theta\ \sin\emptyset & \cos\theta\ \sin\emptyset & \cos\emptyset\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\emptyset\end{bmatrix}$$

$$\begin{bmatrix}\hat{e}_\rho\\ \hat{e}_\varphi\\ \hat{e}_z\end{bmatrix} = \begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sin\theta\cos\emptyset & \cos\theta\cos\emptyset & -\ \sin\emptyset\\ \sin\theta\ \sin\emptyset & \cos\theta\ \sin\emptyset & \cos\emptyset\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\emptyset\end{bmatrix}$$

$$\begin{bmatrix}\hat{e}_\rho\\ \hat{e}_\varphi\\ \hat{e}_z\end{bmatrix} = \begin{bmatrix}\sin\theta & 0 & \cos\theta\\ \cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\emptyset\end{bmatrix}$$

$\therefore\ \hat{e}_\rho = \sin\theta\ \hat{e}_r + \cos\theta\ \hat{e}_\emptyset$

$\hat{e}_\varphi = \cos\theta\ \hat{e}_r - \sin\theta\ \hat{e}_\emptyset$

$\hat{e}_z = \hat{e}_\theta$

Ex. (7): prove that: $\frac{d}{dt}\hat{e}_\rho = \dot{\varphi}\,\hat{e}_\varphi$ and $\frac{d}{dt}\hat{e}_\varphi = -\dot{\varphi}\,\hat{e}_\rho$?

Solution: $\hat{e}_\rho = \cos\varphi\ i + \sin\varphi\ j$, $\hat{e}_\varphi = -\sin\varphi\ i + \cos\varphi\ j$

$$\frac{d}{dt}\hat{e}_\rho = \frac{d}{dt}(\cos\varphi\ i + \sin\varphi\ j) = -\sin\varphi\ \dot{\varphi}\, i + \cos\varphi\ \dot{\varphi}\, j$$

$$= (-\sin\varphi\ i + \cos\varphi\ \ j)\,\dot{\varphi} = \dot{\varphi}\,\hat{e}_\varphi$$

$$\frac{d}{dt}\hat{e}_\varphi = \frac{d}{dt}(-\sin\varphi\ i + \cos\varphi\ j) = -\cos\varphi\ \dot{\varphi}\, i - \sin\varphi\ \dot{\varphi}\, j$$

$$= -(\cos\varphi\ i + \sin\varphi\ j)\,\dot{\varphi} = -\dot{\varphi}\,\hat{e}_\rho$$

Ex. (8): Express the velocity **v** and acceleration **a** of a particle in cylindrical coordinates?

Solution: $x = \rho\cos\varphi$, $y = \rho\ \sin\varphi$, $z = z$

$\therefore i = \cos\varphi\ \hat{e}_\rho - \sin\varphi\ \hat{e}_\varphi$, $j = \sin\varphi\ \hat{e}_\rho + \cos\varphi\ \hat{e}_\varphi$, $k = \hat{e}_z$

$\mathbf{r} = x\, i + y\, j + z\, k$

$= \rho\cos\varphi\left(\cos\varphi\,\hat{e}_\rho - \sin\varphi\ \hat{e}_\varphi\right) + \rho\sin\varphi\left(\sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi\right) + z\,\hat{e}_z$

$\therefore \mathbf{r} = \rho\,\hat{e}_\rho + z\,\hat{e}_z$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\rho}{dt}\hat{e}_\rho + \rho\frac{d\hat{e}_\rho}{dt} + \frac{dz}{dt}\hat{e}_z + z\frac{d\hat{e}_z}{dt} = \dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\varphi}\,\hat{e}_\varphi + \dot{z}\,\hat{e}_z + 0$$

$\therefore \mathbf{v} = \dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\varphi}\,\hat{e}_\varphi + \dot{z}\,\hat{e}_z$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\varphi}\,\hat{e}_\varphi + \dot{z}\,\hat{e}_z\right)$$

$$= \left(\dot{\rho}\ \frac{d\hat{e}_\rho}{dt} + \ddot{\rho}\,\hat{e}_\rho + \rho\,\dot{\varphi}\ \frac{d\hat{e}_\varphi}{dt} + \rho\,\ddot{\varphi}\,\hat{e}_\varphi + \dot{\rho}\,\dot{\varphi}\,\hat{e}_\varphi + \ddot{z}\,\hat{e}_z\right)$$

$$= \left(\dot{\rho}\,\dot{\varphi}\,\hat{e}_\varphi + \ddot{\rho}\,\hat{e}_\rho + \rho\,\dot{\varphi}\left(-\dot{\varphi}\,\hat{e}_\rho\right) + \rho\,\ddot{\varphi}\,\hat{e}_\varphi + \dot{\rho}\,\dot{\varphi}\,\hat{e}_\varphi + \ddot{z}\,\hat{e}_z\right)$$

$$\therefore \mathbf{a} = \left((\ddot{\rho} - \rho\,\dot{\varphi}^2)\,\hat{e}_\rho + (\rho\,\ddot{\varphi} + 2\,\dot{\rho}\,\dot{\varphi})\,\hat{e}_\varphi + \ddot{z}\,\hat{e}_z\right)$$

Ex. (9): Find expressions for the elements of area in orthogonal curvilinear coordinates?

Solution:

$$dA_1 = |(h_2\, du_2\, \hat{e}_2) \times (h_3\, du_3\, \hat{e}_3)| = h_2\, h_3\, |\hat{e}_2 \times \hat{e}_3|\, du_2\, du_3$$

$\therefore dA_1 = h_2\, h_3\, du_2\, du_3$

since $|\hat{e}_2 \times \hat{e}_3| = |\hat{e}_1| = 1$. Similarly

$$dA_2 = |(h_1\, du_1\, \hat{e}_1) \times (h_3\, du_3\, \hat{e}_3)| = h_1\, h_3\, du_1\, du_3$$

$$dA_3 = |(h_1\, du_1\, \hat{e}_1) \times (h_2\, du_2\, \hat{e}_2)| = h_1\, h_2\, du_1\, du_2$$

Ex. (10): Express in cylindrical coordinates the quantities $(\boldsymbol{a})\, \boldsymbol{\nabla}\Phi$, $(\boldsymbol{b})\boldsymbol{\nabla} \,.\, \mathbf{A}$, $(\mathbf{c})\boldsymbol{\nabla} \times \mathbf{A}$, $(\mathbf{d})\boldsymbol{\nabla}^2\Phi$?

Solution: For cylindrical Coordinate $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$

$$u_1 = \rho\,, \quad u_2 = \varphi\,, \quad u_3 = z\,, \quad \hat{e}_1 = \hat{e}_\rho\,, \quad \hat{e}_2 = \hat{e}_\varphi\,, \ \hat{e}_3 = \hat{e}_z$$

$$(\boldsymbol{a})\ \boldsymbol{\nabla}\Phi = \frac{1}{h_1}\frac{\partial \Phi}{\partial u_1}\, \hat{e}_1 + \frac{1}{h_2}\frac{\partial \Phi}{\partial u_2}\, \hat{e}_2 + \frac{1}{h_3}\frac{\partial \Phi}{\partial u_3}\, \hat{e}_3$$

$$= \frac{\partial \Phi}{\partial \rho}\, \hat{e}_\rho + \frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi}\, \hat{e}_\varphi + \frac{\partial \Phi}{\partial z}\, \hat{e}_z$$

$$(\boldsymbol{b})\ \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\,(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}\,(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}\,(h_1 h_2 A_3)\right]$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\,(\rho A_\rho) + \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial}{\partial z}\,(\rho A_z)\right]$$

$$(\mathbf{c})\ \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$= \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\varphi & \hat{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$$

$$(\mathbf{d})\ \nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

<u>Ex. (11):</u> Express in spherical coordinates the quantities $(\boldsymbol{a})\,\nabla\Phi$, $(\boldsymbol{b})\nabla\cdot\mathbf{A}$, $(\mathbf{c})\nabla\times\mathbf{A}$, $(\mathbf{d})\nabla^2\Phi$?

<u>Solution</u>: For spherical Coordinate $h_1 = 1,\ h_2 = r,\ h_3 = r\ \sin\theta$

$u_1 = r,\quad u_2 = \theta,\quad u_3 = \emptyset,\quad \hat{e}_1 = \hat{e}_r,\quad \hat{e}_2 = \hat{e}_\theta,\ \hat{e}_3 = \hat{e}_\emptyset$

$$(\boldsymbol{a})\ \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\ \hat{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\ \hat{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\ \hat{e}_3$$

$$= \frac{\partial\Phi}{\partial r}\ \hat{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\ \hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\emptyset}\ \hat{e}_\emptyset$$

$$(\boldsymbol{b})\ \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\,A_r) + \frac{\partial}{\partial\theta}(r\sin\theta\,A_\theta) + \frac{\partial}{\partial\emptyset}(rA_\emptyset)\right]$$

$$(\mathbf{c})\ \nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\emptyset \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\emptyset} \\ A_r & rA_\theta & r\sin\theta\,A_\emptyset \end{vmatrix}$$

$$(\mathbf{d})\ \nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \emptyset^2}$$

Ex. (12): Express the position vector **r** in spherical coordinates?

Solution: $x = r\ \sin\theta\cos\emptyset\,,\ y = r\ \sin\theta\ \sin\emptyset\,,\ \ z = r\ \cos\theta$

$i = \sin\theta\cos\emptyset\ \hat{e}_r + \cos\theta\cos\emptyset\ \hat{e}_\theta - \ \sin\emptyset\,\hat{e}_\emptyset$

$j = \sin\theta\ \sin\emptyset\ \hat{e}_r + \cos\theta\ \sin\emptyset\ \hat{e}_\theta + \cos\emptyset\ \hat{e}_\emptyset$

$k = \cos\theta\,\hat{e}_r - \sin\theta\ \hat{e}_\theta$

$\mathbf{r} = x\,i + y\,j + z\,k$

$$\begin{aligned}\mathbf{r} = r\ \sin\theta\cos\emptyset\ (\sin\theta\cos\emptyset\ \hat{e}_r + \cos\theta\cos\emptyset\ \hat{e}_\theta - \ \sin\emptyset\,\hat{e}_\emptyset)\\ + r\ \sin\theta\ \sin\emptyset\,(\sin\theta\ \sin\emptyset\ \hat{e}_r + \cos\theta\ \sin\emptyset\ \hat{e}_\theta\\ + \cos\emptyset\ \hat{e}_\emptyset)\ + r\ \cos\theta\,(\cos\theta\,\hat{e}_r - \sin\theta\ \hat{e}_\theta)\end{aligned}$$

$\therefore\ \mathbf{r} = r\,\hat{e}_r \equiv r\,\hat{r}$

Ex. (13): Use spherical coordinates to find **(a)**∇r **(b)**$\nabla\,.\,\mathbf{r}$ **(c)**$\nabla \mathrm{r}^n$ **(d)**$\nabla^2 \mathrm{r}^n$?

Solution: $\because\ \mathbf{r} = r\,\hat{e}_r\quad \Longrightarrow\ this\ functions\ depend\ only\ on\ r$

$$(\mathbf{a})\ \ \nabla r = \hat{e}_r\frac{\partial}{\partial r}r = \hat{e}_r$$

$$(\mathbf{b})\ \ \nabla\,.\,\mathbf{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r) = 3$$

$$(\mathbf{c})\ \nabla \mathrm{r}^n = \hat{e}_r\frac{\partial}{\partial r}\mathrm{r}^n = \hat{e}_r\,n\,\mathrm{r}^{n-1}$$

$$(\mathbf{d})\ \nabla^2 r^n = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}r^n\right) = n\frac{1}{r^2}\frac{\partial}{\partial r}r^{n+1} = n(n+1)\,r^{n-2}$$