1.3.12. Algebra of Logical Proposition

The logical equivalences below are important equivalences that should be memorized.

1-Identity Laws: $p \wedge T \equiv p.$
$p \vee F \equiv p.$

2-Domination Laws: $p \vee T \equiv T.$
$p \wedge F \equiv F.$

3-Idempotent Laws: $p \vee p \equiv p.$
$p \wedge p \equiv p.$

4- Double Negation Law: $\sim(\sim p) \equiv p.$

5- Commutative Laws: $p \vee q \equiv q \vee p.$
$p \wedge q \equiv q \wedge p.$

6- Associative Laws: $(p \vee q) \vee r \equiv p \vee (q \vee r).$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r).$

7- Distributive Laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$

8- De Morgan's Laws: $\sim (p \wedge q) \equiv \sim p \vee \sim q.$
$\sim (p \vee q) \equiv \sim p \wedge \sim q.$

9- Absorption Laws: $p \wedge (p \vee q) \equiv p.$
$p \vee (p \wedge q) \equiv p.$
$p \wedge (\sim p \vee q) \equiv p \wedge q.$
$p \vee (\sim p \wedge q) \equiv p \vee q.$

10-Implication Law: $(p \rightarrow q) \equiv (\sim p \vee q).$

11- Contrapositive Law: $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p).$

12- Tautology: $p \vee \sim p \equiv T.$

13- Contradiction: $p \wedge \sim p \equiv F.$

14- Equivalence: $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q).$

15- $p \underline{\vee} q \equiv (p \vee q) \wedge \sim(p \wedge q).$

Solution.

(8) We using truth table to prove $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(14) We using truth table to prove $(p \to q) \wedge (q \to p) \equiv (p \leftrightarrow q)$.

p	q	$p \to q$	$q \to p$	$p \to q \wedge q \to p$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

(15) $p \underline{\vee} q \equiv (p \vee q) \wedge \sim (p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$\mathbf{p \underline{\vee} q}$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	T	F	F

1.4. Rules of Proof

1.4.1.

(i) Rule of Replacement.

Any term in a logical formula may be replaced be an equivalent term.

For instance, if $q \equiv r$, then $p \wedge q \equiv p \wedge r$ Rep(q:r).

(ii) Rule of Substitution.

A sentence which is obtained by substituting logical propositions for the terms of a theorem is itself a theorem.

For instance, $(p \to q) \vee w \equiv w \vee (p \to q)$ Sub(p: $p \to q$), Theorem $p \vee w \equiv w \vee p$.

(iii) Rule of Inference.

p $\underline{p \to q}$ $\therefore q$	$\sim q$ $\underline{p \to q}$ $\therefore \sim p$
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$p \vee q$ $\underline{\sim p}$ $\therefore q$
$\underline{p}$ $\therefore p \vee R$	$\underline{p \wedge q}$ $\therefore p$
p $\underline{q}$ $\therefore p \wedge q$	$p \vee q$ $\underline{\sim p \vee r}$ $\therefore q \vee r$
p $\underline{q \to r}$ $\therefore p \vee q \to p \vee r$	$p \to q$ $\underline{r \to t}$ $\therefore p \vee r \to q \vee t$

Example 1.4.2. Given
(1) "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"
(2) "If the sailing race is held, then the cup will be awarded"
(3) "The cup was not awarded"
Does this imply that: "It rained"?
Solution.
p: rain
q: foggy
r: the sailing race will be held
s: the lifesaving demonstration will go on
t: then the cup will be awarded

Symbolically, the proposition is

$$\begin{array}{ll} (1) & \sim p \vee \sim q \;\rightarrow\; r \wedge s \\ (2) & r \;\rightarrow\; t \\ (3) & \sim t \\ \hline & p \end{array}$$

1. $\sim t$ 3rd hypothesis
2. $r \rightarrow t$ 2nd hypothesis
3. $\sim t \rightarrow \sim r$ Contrapositive of 2
4. $\sim r$ inf (1),(3)
5. $\sim p \vee \sim q \rightarrow r \wedge s$ 1st hypothesis
6. $\sim (r \wedge s) \rightarrow \sim (\sim p \vee \sim q)$ Contrapositive of 5
7. $\sim r \vee \sim s \rightarrow (p \wedge q)$ De Morgan's law and double negation law from 5
8. $\sim r \vee \sim s$ inf (4) and domination law
9. $p \wedge q$ inf (7),(8)
10. p inf (9)

Example 1.4.3. Use the logical equivalences to show that
(i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$,
(ii) $\sim(p \vee \sim(p \wedge q))$ is a contradiction,
(iii) $\sim (p \vee (\sim p \wedge q)) \equiv (\sim p \wedge \sim q)$,

(iv) $p \vee (p \wedge q) \equiv p$ (Absorption Law).

Solution.

(i) $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$ Implication Law

$\equiv \sim(\sim p) \wedge \sim q.$ De Morgan's Law
$\equiv p \wedge \sim q$ Double Negation Law

(ii) $\sim(p \vee \sim(p \wedge q))$
$\equiv \sim p \wedge \sim(\sim(p \wedge q))$ De Morgan's Law
$\equiv \sim p \wedge (p \wedge q)$ Double Negation Law
$\equiv (\sim p \wedge p) \wedge q$ Associative Law
$\equiv F \wedge q$ Contradiction Law
$\equiv F$ Domination Law and Commutative Law.

(iii) $\sim(p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim(\sim p \wedge q)$ De Morgan's Law

$\equiv \sim p \wedge (\sim\sim p \vee \sim q)$ De Morgan's Law

$\equiv \sim p \wedge (p \vee \sim q)$ Double Negation Law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ Distribution Law

$\equiv (p \wedge \sim p) \vee (\sim p \wedge \sim q)$ Commutative Law

$\equiv F \vee (\sim p \wedge \sim q)$ Contradiction Law

$\equiv (\sim p \wedge \sim q) \vee F$ Commutative Law

$\equiv (\sim p \wedge \sim q)$ Identity Law

(iv) $p \vee (p \wedge q)$
$\equiv (p \wedge T) \vee (p \wedge q)$ Identity (in reverse)
$\equiv p \wedge (T \vee q)$ Distributive (in reverse)
$\equiv p \wedge T$ Domination
$\equiv p$ Identity

Example 1.4.4. Find a simple form for the negation of the proposition
"If the sun is shining, then I am going to the ball game."
Solution.

p: the sun is shining q: I am going to the football game

This proposition is of the form $p \to q$. Since $\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv (p \wedge \sim q)$.This is the proposition "The sun is shining, and I am not going to the football game."

1.5. Normal or Canonical Forms

From Examples 1.4.1.(ii)(iii), even though are expressed with only $\wedge$, $\vee$ and $\sim$, it is still hard to tell without doing a proof.

(i) What we need is a unique representation of a compound proposition that uses $\wedge$, $\vee$ and $\sim$.

(ii) This unique representation is called the Disjunctive Normal Form as define below.

Definition 1.5.1. A clause that contains only $\vee$ is called a **disjunctive clause** and only $\wedge$ is called a **conjunctive clause**.

Negation is allowed, but only directly on variables.

Example 1.5.2.

(i) $p\vee\sim q\vee r$: a disjunctive clause.

(ii) $\sim p\wedge q\wedge\sim r$: a conjunctive clause.

(iii) $\sim p\wedge\sim q\vee r$: neither.

Definition 1.5.3.

(i) A bunch of disjunctive clauses together with $\wedge$, it is called **conjunctive normal form(CNF)**.

(i) A bunch of conjunctive clauses together with $\vee$, it is called **disjunctive normal form(DNF)**.

Remark 1.5.4. The individual conjunction clauses (disjunctive clauses) that make up the DNF (CNF) are called **minterms**.

Example 1.5.5.

(i) p : Both normal forms, since single literal is always conjunctive and disjunctive normal form.

(ii) $p \wedge q$: Conjunctive and disjunctive normal form.

(iii) $(p\wedge q\wedge\sim r \wedge s)\vee(\sim q\wedge s)\vee(p\wedge s)$: disjunctive normal form.

(iv) $(p\vee q\vee\sim r\vee s) \wedge (\sim q\vee s)\wedge \sim s$: conjunctive normal form.

(v) $(p\vee r)\wedge(q\wedge(p\vee\sim q))$: not in a normal form.

(vi) $\sim p \vee q \vee r$ and $\sim p \wedge q \wedge r$: Both normal forms.

Remark 1.5.6. It turns out we can turn any logical proposition into either normal form.

(i) We can use the definitions to get rid of $\rightarrow$, $\leftrightarrow$, and $\underline{\vee}$.

(ii) Use De Morgan's laws to move any $\sim$ in past original statement, so they sit on the variables.

(iii) Use double negation to get rid of any $\sim\sim$ that showed up.

(iv) Use the distributive rules to move things in/out of original statement as we need to.

Example 1.5.7. Converting $\sim((\sim p\rightarrow\sim q)\wedge \sim r)$ to conjunctive normal form.

$\sim((\sim p\rightarrow\sim q)\wedge \sim r)$	$\equiv \sim((\sim\sim p\vee\sim q)\wedge \sim r)$	Definition
	$\equiv \sim((p\vee\sim q)\wedge \sim r)$	Double negation
	$\equiv \sim(p\vee\sim q)\vee \sim\sim r$	De Morgan's
	$\equiv\sim (p\vee\sim q)\vee r$	Double negation
	$\equiv(\sim p\wedge\sim\sim q)\vee r$	De Morgan's
	$\equiv(\sim p\wedge q)\vee r$	Double negation : disjunctive normal form
	$\equiv(\sim p\vee r)\wedge(q\vee r)$	Distributive : conjunctive normal form

It was actually in disjunctive normal form in the second-last step.

1.6. Method To Construct DNF

To construct DNF of a logical proposition we use the following way.

Construct a truth table for the proposition.

(i) Use the rows of the truth table where the proposition is True to construct minterms

– If the variable is true, use the propositional variable in the minterm.

– If a variable is false, use the negation of the variable in the minterm.

(ii) Connect the minterms with $\vee$'s.

Example 1.6.1. Find the disjunctive normal form for the following logical proposition
(i) $p \rightarrow q$.
(ii) $(p \rightarrow q) \wedge \sim r$.
Solution. (i) Construct a truth table for $p \rightarrow q$:

p	q	p → q	
T	T	T	←
T	F	F	
F	T	T	←
F	F	T	←

$p \rightarrow q$ is true when either
p is true and q is true, or
p is false and q is true, or
p is false and q is false.
The disjunctive normal form is then

$$(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q).$$

(ii) Write out the truth table for $(p \rightarrow q) \wedge \sim r$

p	q	r	p → q	~ r	(p → q) ∧ ~ r	
T	T	T	T	F	F	
T	T	F	T	T	T	←
T	F	T	F	F	F	
T	F	F	F	T	F	
F	T	T	T	F	F	
F	T	F	T	T	T	←
F	F	T	F	F	F	
F	F	F	T	T	T	←

The disjunctive normal form for $(p \to q) \wedge \sim r$ is

$$(p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r).$$

Remark 1.6.2. If we want to get the conjunctive normal form of a logical proposition, construct
(1) the disjunctive normal form of its negation,
(2) negate again and apply De Morgan's Law.
Example 1.6.3. Find the conjunctive normal form of the logical proposition

$$(p \wedge \sim q) \vee r.$$

Solution.
(1) Negate: $\sim[(p \wedge \sim q) \vee r] \equiv (\sim p \vee q) \wedge \sim r$.

(2) Find the disjunctive normal form of $(\sim p \vee q) \wedge \sim r$.

p	q	r	$\sim p$	$\sim r$	$\sim p \vee q$	$(\sim p \vee q) \wedge \sim r$	
T	T	T	F	F	T	F	
T	T	F	F	T	T	T	←
T	F	T	F	F	F	F	
T	F	F	F	T	F	F	
F	T	T	T	F	T	F	
F	T	F	T	T	T	T	←
F	F	T	T	F	T	F	
F	F	F	T	T	T	T	←

The disjunctive normal form for $(\sim p \vee q) \wedge \sim r$ is
DNF: $(p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)$.

(3) The conjunctive normal form for $(p \wedge \sim q) \vee r$ is then the negation of this last expression, which, by De Morgan's Laws, is
CNF: $(\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee r)$.

Remark 1.6.4.

(1) $p \vee q$ can be written in terms of $\wedge$ and $\sim$ since $p \vee q \equiv (\sim\sim p \vee \sim\sim q) \equiv \sim(\sim p \wedge \sim q)$.

(2) We can write every compound logical proposition in terms of $\wedge$ and $\sim$.

1.7. Logical Implication

Definition 1.7.1. (Logical implication)

We say the logical proposition r implies the logical proposition s (or s logically deduced from r) and write $r \Longrightarrow s$ if $r \to s$ is a tautology.

Example 1.7.2. Show that $(p \to t) \wedge (t \to q) \Longrightarrow p \to q$.

Solution. Let P: the proposition $(p \to t) \wedge (t \to q)$

Q: the proposition $p \to q$

p	t	q	$p \to t$	$t \to q$	P	Q	$P \to Q$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Remark 1.7.3.

(i) We use $r \Longrightarrow s$ to imply that the statement $r \to s$ is true, while the statement $r \to s$ alone does not imply any particular truth value. The symbol is often used in proofs as shorthand for "implies".

(ii) If $r \Longrightarrow s$ and $s \Longrightarrow r$, then we denote that by $r \Longleftrightarrow s$.

Example 1.7.4. Show that

(i) $r \Longrightarrow s$ if and only if $\sim r \vee s$ is tautology.
(ii) $r \Longleftrightarrow s$ if and only if $r \equiv s$.

Solution.

(i) $r \Longrightarrow s$ if and only if $r \to s$ is a tautology (by def.)
But $\sim r \vee s \equiv r \to s$ is a tautology.
Then, $r \Longrightarrow s$ if and only if $\sim r \vee s$ is tautology.

(ii) $r \Longrightarrow s$ if and only if $r \to s$ is tautology (by def.)
$s \Longrightarrow r$ if and only if $s \to r$ is tautology (by def.)
Then, $r \to s \wedge s \to r$ is tautology.
Therefore, $r \equiv s$.

- Generally, the statement and its converse not necessary equivalent. Therefore, $p \Longrightarrow q$ does not mean that $q \Longrightarrow p$.

Example 1.7.5. The statement "the triangle which has equal sides, has two equal legs" equivalent to the statement " the triangle which has not two equal legs has no equal sides".