

# Chapter Four

## Growth of Cloud Droplets by Diffusion

### Flux

- ) A flux is the amount of something passing through a unit of area in a unit of time.
- ) The units of flux are the units of whatever is being transported, divided by area and time. Examples are:
  - Mass flux:**  $\text{kg m}^{-2} \text{s}^{-1}$
  - Energy flux:**  $\text{J m}^{-2} \text{s}^{-1}$
  - Particle flux:**  $\text{m}^{-2} \text{s}^{-1}$
- ) The flux is actually a vector that points in the direction of the transport.
- ) In component form in Cartesian coordinates the flux vector is:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \quad (4.1)$$

### Growth Rate of Droplet by Diffusion

- ) Once a cloud droplet forms it continues to grow by diffusion of water vapor onto its surface (condensation).
- ) Figure 1 illustrates a droplet of radius  $R$  with radial vapor fluxes at the surface of the droplet denoted by  $\vec{F}_R$

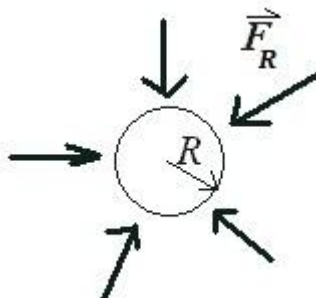


Figure 1: Convergence of radial vapor fluxes  $\vec{F}_R$  at the surface of the droplet results in droplet growth.

) For simplicity we will assume that the fluxes are *axisymmetric*, meaning that the fluxes only change with distance from the droplet, not with the angle. Another way of saying this is that the fluxes are *isotropic*.

) If we multiply the flux at the surface of the droplet by the surface area of the droplet we obtain the rate of change of molecules of the droplet,

$$\frac{dn}{dt} = 4\pi R^2 F_R \quad (4.2)$$

- Note that  $F_R$  itself is negative, since it is pointing inward toward the droplet. That is why there is a negative in front of (4.2), so that  $dn/dt$  will be positive.

) The flux  $F_R$  at the surface of the droplet is given by Fick's first law of diffusion  $\vec{F} = -D \nabla N$  where  $D$  is the *diffusivity*, and is  $F_R = -\hat{k} \cdot D (\nabla N)_R = -D \left( \frac{dN}{dr} \right)_R$ . Therefore (4.2) becomes:

$$\frac{dn}{dt} = 4\pi R^2 D \left( \frac{dN}{dr} \right)_R \quad (4.3)$$

- Keep in mind that  $n$  is the number of water molecules in the droplet itself, whereas  $N$  is the number density of water vapor molecules in the air.

) We find  $\left( \frac{dN}{dr} \right)_R$  as follows:

- We assume that  $N$  does not change with time, so that from Fick's second law of diffusion  $\frac{d}{dt} \left( \frac{dN}{dr} \right) = 0$  we have:

$$\frac{d}{dr} \left( r^2 \frac{dN}{dr} \right) = 0 \quad (4.4)$$

- In spherical coordinates with the droplet at the origin, and since the vapor concentration is axisymmetric, (4.4) becomes

$$\frac{d}{dr} \left( r^2 \frac{dN}{dr} \right) = 0 \quad (4.5)$$

- Integrating (4.5) twice with respect to  $r$  results in

$$N(r) = \frac{c_1}{r} + c_2 \quad (4.6)$$

where  $c_1$  and  $c_2$  are the constants of integration. We find them by applying the boundary conditions

$$\begin{aligned} N(r \gg R) &= N_b \\ N(R) &= N_R \end{aligned} \quad (4.7)$$

where  $N_b$  is the background vapor concentration well away from the droplet.

- Applying the boundary conditions (4.7) to (4.6) results in

$$\begin{aligned} c_1 &= N_b Z N_R R \\ c_2 &= N_b \end{aligned}$$

- Putting these constants back into (4.6) results in

$$N(r) = N_b Z \frac{(N_b Z N_R) R}{r} \Gamma N_b \quad (4.8)$$

- And finally, by taking  $\frac{dN}{dr}$  of (4.8) and evaluating the result at  $r = R$ , we get

$$\left. \frac{dN}{dr} \right|_{r=R} = -N_b Z \frac{N_b Z N_R}{R} \quad (4.9)$$

- ) Putting (4.9) into (4.3) gives us our growth-rate equation for the droplet,

$$\frac{dn}{dt} = 4\pi R^2 D (N_b - N_R) \quad (4.10)$$

- If the background vapor concentration is larger than that at the droplet surface,  $N_b > N_R$ , the droplet will grow due to condensation.
- If the background vapor concentration is smaller than that at the droplet surface,  $N_b < N_R$ , the droplet will shrink due to evaporation.

## Growth Rate in Terms of Droplet Mass and Radius

- ) Equation (4.10) can be converted to an equation for the mass growth rate,  $dm/dt$ , as follows:

- Multiply both sides of (4.10) by the molar mass of water,  $M_w$ , and divide by Avogadro's number,  $N_A$ ,

$$\frac{M_w}{N_A} \frac{dn}{dt} = 4\pi R^2 D \frac{M_w}{N_A} (N_b - N_R) \quad (4.11)$$

- Since mass is

$$\frac{M_w}{N_A} n = m$$

and absolute humidity is

$$\frac{M_w}{N_A} N \sum \partial_v$$

(4.11) becomes

$$\frac{dm}{dt} \sum \partial_v R (\partial_{v_b} \sum \partial_{v_R}) \quad (4.12)$$

) What would be most convenient is to have an equation for the growth-rate in terms of the radius of the droplet. We can construct this using the chain rule for derivatives,

$$\frac{dR}{dt} \sum \frac{dR}{dm} \frac{dm}{dt} \quad (4.13)$$

- The mass of a droplet is

$$m \sum \frac{4}{3} \pi R^3$$

so

$$\frac{dR}{dm} \sum \frac{1}{4 \pi R^2} \quad (4.14)$$

) From (4.12), (4.13) and (4.14) we get

$$R \frac{dR}{dt} \sum \frac{D}{\partial_i} (\partial_{v_b} \sum \partial_{v_R}) \quad (4.15)$$

## Other Equations Needed to Solve for Growth Rate

) Equation (4.15) gives us the ability to integrate forward in time to find an expression for  $R(t)$ , the radius of the droplet at any future time  $t$ .

- We do not know what value of  $v_R$  to use, since this depends on the temperature of the surface of the droplet.
- However, we can assume that at the surface of the droplet the air is saturated, so that  $v_R = v_s$ , where  $v_s$  is the *saturation absolute humidity*.
- From the ideal gas law for water vapor

$$\partial_{v_R} \sum \partial_{v_s} \sum \frac{e_s}{R_v T_R} \quad (4.16)$$

where  $T_R$  is the temperature at the surface of the droplet.

\* Note that  $T_R$  is not necessarily the same as the air temperature. The droplet warms or cools depending on whether there is condensation or evaporation at the droplet's surface.

- $e_s$  is the saturation vapor pressure over a curved, impure droplet which we know to be

$$e_s = X e_o \left[ 1 - \Gamma \frac{a}{R} \right] \frac{b}{R^3} \exp \left[ \frac{L_v}{R_v} \left( \frac{1}{T_o} - \frac{1}{T_R} \right) \right] \quad (4.17)$$

so that

$$\partial_{vR} \left[ X \frac{e_o}{R_v T_R} \left[ 1 - \Gamma \frac{a}{R} \right] \frac{b}{R^3} \exp \left[ \frac{L_v}{R_v} \left( \frac{1}{T_o} - \frac{1}{T_R} \right) \right] \right] \quad (4.18)$$

) Equations (4.15) and (4.18) are two equations, but we have three unknown quantities:  $R$ ,  $vR$ , and  $T_R$ . therefore we still need one more equation in order to have a closed set that we can solve.

) The third equation comes from balancing the gain of latent heat due to condensation with the loss of sensible heat due to thermal diffusivity.

- The gain of latent heat due to condensation is given by

$$J_{latent} = X L_v \frac{dm}{dt} = 4 \pi R^2 D (\partial_{vb} - Z \partial_{vR}) \quad (4.19)$$

- The sensible lost to the air by diffusion is

$$J_{sensible} = X Z 4 \pi R K (T_R - T_b) \quad (4.20)$$

where  $K$  is the thermal diffusivity of air and  $T_b$  is the temperature of the air.

- Balancing the sensible and latent heats by setting (4.19) equal to (4.20) results in

$$\partial_{vb} - Z \partial_{vR} = X \frac{K}{L_v D} (T_R - T_b) \quad (4.21)$$

## Calculations of Growth Rates

) Equations (4.15), (4.18) and (4.21) are three equations for three unknown quantities,  $R$ ,  $vR$ , and  $T_R$ . The equations are rewritten here,

$$R \frac{dR}{dt} = X \frac{D}{\partial_l} (\partial_{vb} - Z \partial_{vR})$$

$$\partial_{vR} \left[ X \frac{e_o}{R_v T_R} \left[ 1 - \Gamma \frac{a}{R} \right] \frac{b}{R^3} \exp \left[ \frac{L_v}{R_v} \left( \frac{1}{T_o} - \frac{1}{T_R} \right) \right] \right]$$

$$\partial_{vb} - Z \partial_{vR} = X \frac{K}{L_v D} (T_R - T_o)$$

- ) We can solve these three equations to find the growth rate and radius of a droplet at any future time,  $t$ .
- ) However, the equations are quite complex and cannot be solved analytically. They need to be solved numerically.
- ) A somewhat simplified, though not as accurate, set of growth equations is

$$R \frac{dR}{dt} \times \frac{S \Gamma \frac{a}{R} \Gamma \frac{b}{R^3}}{F_k \Gamma F_d} \quad (4.22)$$

$$F_k \times \frac{L_v}{R_v T_b} \Gamma \frac{L_v \partial_l}{K T_b} \quad (4.23)$$

$$F_d \times \frac{\partial_l R_v T_b}{D e_{s|}^*} \quad (4.24)$$

where the saturation vapor pressure used in calculating  $F_d$  is that for a flat surface of pure water

- ) These equations still need to be integrated numerically. The result for a droplet starting at radius  $r_o = 0.75 \mu\text{m}$  is shown in Figure 2.
- ) Note that after 20 hours the droplet is still only has a radius slightly larger  $60 \mu\text{m}$ .
- ) Figure 3 shows the effect of doubling the mass of solute. Although the droplet initially grows faster with more solute, the growth rates quickly become the same.

## Final Comments on Diffusional Growth

- ) In order to be large enough to fall fast enough to reach the ground without evaporating, a droplet has to reach a size of at least 0.1 mm in diameter (0.05 mm or 50  $\mu\text{m}$  in radius).
- ) A typical raindrop has a diameter of 2 mm (radius of 1 mm, or 1000  $\mu\text{m}$ ).
- ) Clouds can form and rain start to fall in a matter of 30 minutes or so.
- ) Diffusional growth explains how very tiny, brand-new cloud droplets grow to typical cloud droplet sizes, but is too slow to explain how precipitation forms.

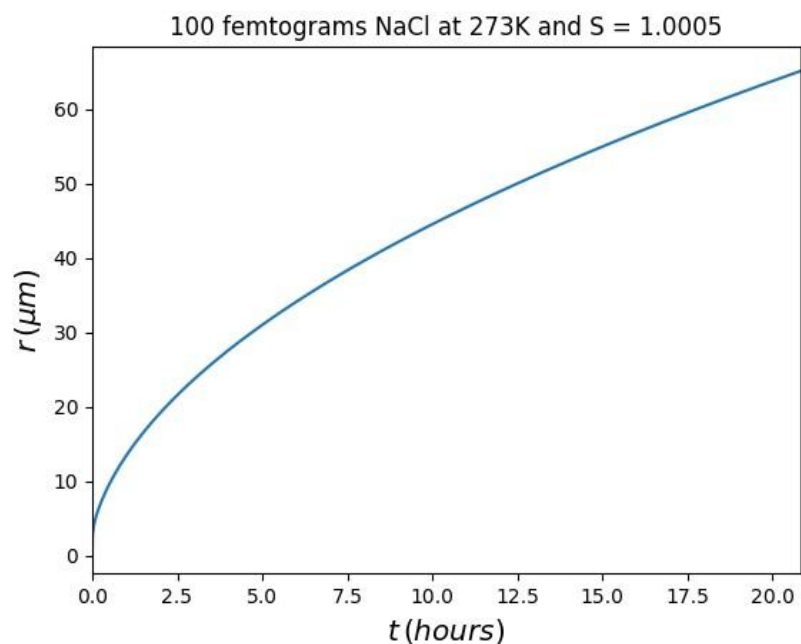


Figure 4: Growth of droplet initially of radius  $0.75 \mu\text{m}$  for a solute of 100 femtograms of NaCl.

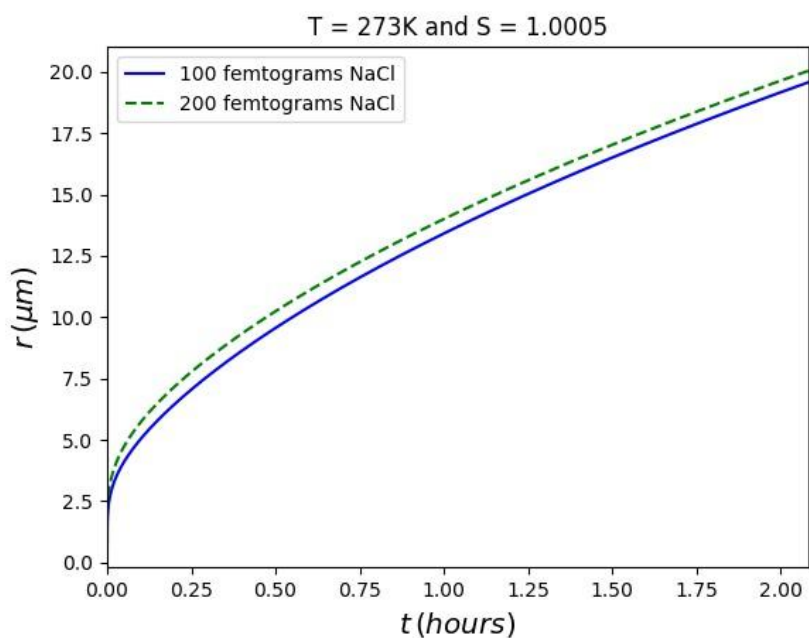


Figure 5: Growth of droplet initially of radius  $0.75 \mu\text{m}$  for two different solute masses.