

Lecture 10

The Spherical Coordinates

10.1 Component Equations in Spherical Coordinates

It is convenient to expand the momentum equation of motion in spherical coordinates. The coordinate axes are then (λ, ϕ, z) , where λ is longitude, ϕ is latitude, and z is the vertical distance above the surface. If the unit vectors $i, j,$ and k are now taken to be directed eastward, northward, and upward, respectively. Then:

$$\vec{V} \equiv iu + jv + kw$$

where u, v and w are defined as:

$$u \equiv r \cos \phi \frac{d\lambda}{dt}, \quad v \equiv c \frac{d\phi}{dt}, \quad w \equiv \frac{dz}{dt} \quad (10.1)$$

Here, r is the distance to the center of earth, which is related to z by $r = a + z$, where a is the radius of the earth. Traditionally, the variable r in (10.1) is replaced by the constant a for notational simplicity (Why?).

It is conventional to define x and y as eastward and northward distance such that:

$$dx = a \cos \phi d\lambda \quad \text{and} \quad dy = a d\phi$$

Thus, $u \equiv \frac{dx}{dt}$ and $v \equiv \frac{dy}{dt}$ are the horizontal velocity components.

The (x, y, z) coordinate system defined in this way is not a Cartesian coordinate system because the directions of the i, j, k unit vectors are not constant as function of position on the spherical earth. Thus:

$$\frac{d\vec{V}}{dt} = i \frac{du}{dt} + j \frac{dv}{dt} + k \frac{dw}{dt} + u \frac{di}{dt} + v \frac{dj}{dt} + w \frac{dk}{dt} \quad (10.2)$$

Now, we first consider $\frac{di}{dt}$ and by expanding the total derivative and noting that i is a function only of x :

$$\frac{di}{dt} = u \frac{\partial i}{\partial x}$$

(1 - 3)

From Figure (10.1) we see by similarity of triangles:

$$\lim_{\Delta x \rightarrow 0} \frac{|\Delta i|}{\Delta x} = \left| \frac{\partial i}{\partial x} \right| = \frac{1}{a \cos \phi}$$

and that the vector $\frac{\partial i}{\partial x}$ is directed toward the

axis of rotation. Thus, as is illustrated in Fig (10.2):

$$\frac{\partial i}{\partial x} = \frac{1}{a \cos \phi} (j \sin \phi - k \cos \phi)$$

Therefore:

$$\frac{di}{dt} = \frac{u}{a \cos \phi} (j \sin \phi - k \cos \phi) \quad (10.3)$$

Considering now $\frac{dj}{dt}$, we note that j is a function only of x and y . Thus, with the aid of Fig. (10.3) we see that for eastward motion $|\Delta j| = \Delta x / (a / \tan \phi)$.

Because the vector $\frac{dj}{dx}$ is directed in the negative x direction,

We have then:

$$\frac{dj}{dx} = -\frac{\tan \phi}{a} i$$

From Fig. 10.4 it is clear that for northward motion $|\Delta j| = \Delta \phi$, and $\Delta y = a \Delta \phi$ and Δj is directed downward so that:

$$\frac{dj}{dy} = -\frac{k}{a}$$

Hence,

$$\frac{dj}{dt} = -\frac{u \tan \phi}{a} i - \frac{v}{a} k \quad (10.4)$$

Finally, by similar arguments it can be shown that:

$$\frac{dk}{dt} = i \frac{u}{a} + j \frac{v}{a} \quad (10.5)$$

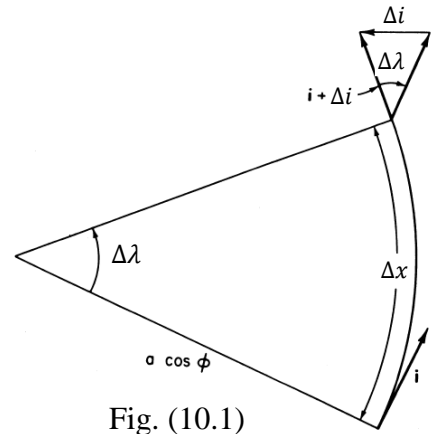


Fig. (10.1)

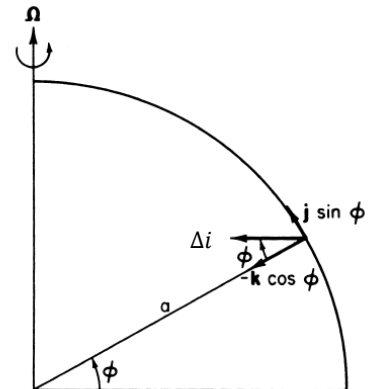


Fig. (10.2) Resolution of Δi in Fig. 9.1 into northward and vertical components

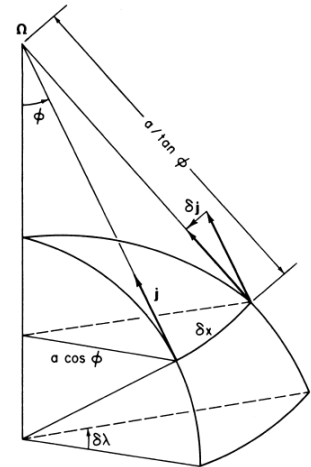


Fig. (10.3) Dependence of j on longitude

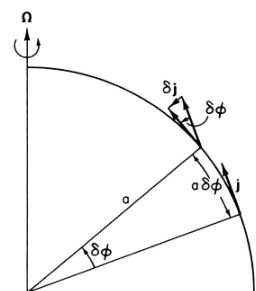


Fig. (10.4) Dependence of j on latitude

Substituting (10.3)-(10.5) into (10.2) and rearranging the terms,

We obtain the spherical polar coordinate expansion of the acceleration following the relative motion:

$$\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} \right) i + \left(\frac{dv}{dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} \right) j + \left(\frac{dw}{dt} - \frac{u^2 + v^2}{a} \right) k \quad (10.6)$$

From Equation (9.1) in Lecture (9):

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho} \nabla p + \vec{g} + \vec{F}_r - 2\vec{\Omega} \times \vec{V}$$

By substituting the left side from equation (10.6) and the right side terms from lecture (9) and decomposed the equation into the three directions:

$$\frac{du}{dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial z} \right) + 2\Omega w \cos \phi - 2\Omega v \sin \phi \quad (10.7)$$

$$\frac{dv}{dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{1}{\rho} \frac{\partial}{\partial z} \left(\mu \frac{\partial v}{\partial z} \right) - 2\Omega u \sin \phi \quad (10.8)$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \frac{1}{\rho} \frac{\partial}{\partial z} \left(\mu \frac{\partial w}{\partial z} \right) - g + 2\Omega u \cos \phi \quad (10.9)$$

Notes:

1. The terms proportional to $1/a$ on the left side of momentum equations above are called curvature terms; they arise due to the curvature of the earth. Because they are nonlinear (i.e., they are quadratic in the dependent variable), they are difficult to handle in theoretical analyses. Fortunately, the curvature terms are unimportant for midlatitude synoptic scale motions.
2. Even if we neglect the curvature terms, the equations still nonlinear. This is obvious when we expand the total derivative $\left(\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$.
3. The presence of nonlinear advection processes is one reason that dynamic meteorology is an interesting and challenging subject.