FOUNDATION OF MATHEMATICS I

CHAPTER THREE (CROSS PRODUCT AND RELATIONS)

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Definition 3.2.9. R is **a totally order** on A if R is a partial order, and xRy or yRx for all $x, y \in A$; that is, if any two elements of A are comparable with respect to R. Then we call the pair $(A, \leq)$ a **totally order set** or a **chain**.

Example 3.2.10.
(i) Let $A = \{2, 3, 4, 5, 6\}$, and define R by the usual $\leq$ relation on $\mathbb{N}$, i.e. aRb iff $a \leq b$. Then R is a **totally order** on A.
(ii) Let us define another relation on $\mathbb{N}$

$$a/b \text{ iff } a \text{ divides } b.$$

To show that / is a partial order we have to show the three defining properties of a partial order relation: **Reflexive:** Since every natural number is a divisor of itself, we have a/a for all $a \in A$.
Antisymmetric: If a divides b then we have either $a = b$ or $a < b$ in the usual ordering of $\mathbb{N}$; similarly, if b divides a, then $b = a$ or $b < a$. Since $a < b$ and $b < a$ is not possible, a/b and b/a implies $a = b$.
Transitive: If a divides b and b divides c then a also divides c. Thus, / is a partial order on N.
(iii) Let $A = \{x, y\}$ and define $\leq$ on the power set $P(A)$ by

$$s \leq t \text{ iff } s \text{ is a subset of } t.$$

This gives us the following relation:
$\emptyset \leq \emptyset, \emptyset \leq \{x\}, \emptyset \leq \{y\}, \emptyset \leq \{x, y\} = A, \{x\} \leq \{x\}, \{x\} \leq \{x, y\}, \{y\} \leq \{y\}, \{y\} \leq \{x, y\}, \{x, y\} \leq \{x, y\}$.
Exercise 3.2.11.
Let $A = \{1, 2, \ldots, 10\}$ and define the relation R on A by xRy iff x is a multiple of y. Show that R is a partial order on A.
Definition 3.2.12. (Inverse of a Relation)
Suppose $R \subseteq A \times B$ is a relation between A and B then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by

$$bR^{-1}a \quad \text{if and only if} \quad aRb.$$

That is, $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

Example 3.2.13. Let R be the relation between $\mathbb{Z}$ and $\mathbb{Z}^+$ defined by

$$mRn \text{ if and only if } m^2 = n.$$

Then

$$R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z}^+ : m^2 = n\},$$

and

$$R^{-1} = \{(n,m) \in \mathbb{Z}^+ \times \mathbb{Z} : m^2 = n\}.$$

For example, –3 R 9, –4 R 16, 16 R^{-1} 4, 9 R^{-1} 3, etc.

Remark 3.2.14.

If R is partial order relation on $A \neq \emptyset$, then R^{-1} is also partial order relation on A.

Proof.

(i) Reflexive. Let $x \in A$.

$\Longrightarrow (x,x) \in R$ (Reflexivity of A) $\Longrightarrow (x,x) \in R^{-1}$ Def of R^{-1}

(ii) Antisymmetric. Let $(x,y) \in R^{-1}$ and $(y,x) \in R^{-1}$. To prove $x = y$.

$\Longrightarrow (y,x) \in R \wedge (x,y) \in R$ Def of R^{-1}

$\Longrightarrow y = x$ Since R is antisymmetric

(iii) Transitive. Let $(x,y) \in R^{-1}$ and $(y,z) \in R^{-1}$. To prove $(x,z) \in R^{-1}$.

$\Longrightarrow (y,x) \in R \wedge (z,y) \in R$ Def of R^{-1}

$\Longrightarrow (z,y) \in R \wedge (y,x) \in R$ Commut. Law of $\wedge$

$\Longrightarrow (z,x) \in R$ Since R is transitive

$\Longrightarrow (x,z) \in R^{-1}$ Def of R^{-1}

Definition 3.2.15. (Partitions)

Let A be a set and let $A_1, A_2, \ldots, A_n$ be subsets of A such

(i) $A_i \neq \emptyset$ for all i,

(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$,

(iii) $A = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$. Then the sets A_i partition the set A and these sets are called the **classes of the partition.**

Remark 3.2.16. An equivalence relation on X leads to a partition of X, and **vice versa** for every partition of X there is a corresponding equivalence relation.

Proof:

(a) Let R be an equivalence relation on X.

1- $\forall a \in X, a \in [a]$ Def. of equ. Class

2- $\exists [b] \in X/R$ such that $[b] = [a]$ Since X/R contains all diff. classes

3- $X = \bigcup_{a \in X}\{a\} \subseteq \bigcup_{a \in X}[a] \subseteq \bigcup_{a \in [b]}[b] \subseteq X \Longrightarrow X = \bigcup_{[b] \in X/R}[b]$.

4- $[b] \cap [a] = \emptyset$, for all $[b],[a] \in X/R$ Def. of X/R

5- R is partition of X Inf.(3),(4)

(b) Let (i) $A_i \neq \emptyset$ for all i, $A_i \subseteq X$

(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$,

(iii) $X = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$.

Define R(relation) on X by aRb if $\exists\, A_i, A_j$ such that $a \in A_i \wedge b \in A_j$, $i \neq j$.

This relation is an equivalence relation on X.

Definition 3.2.17. (The Composition of Two Relations)

The composition of two relations $R_1(A,B)$ and $R_2(B,C)$ is given by $R_2\ o\ R_1$ where $(a,c) \in R_2\ o\ R_1$ if and only if there exists $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$.

Remark 3.2.18. The composition of relations is associative; that is,

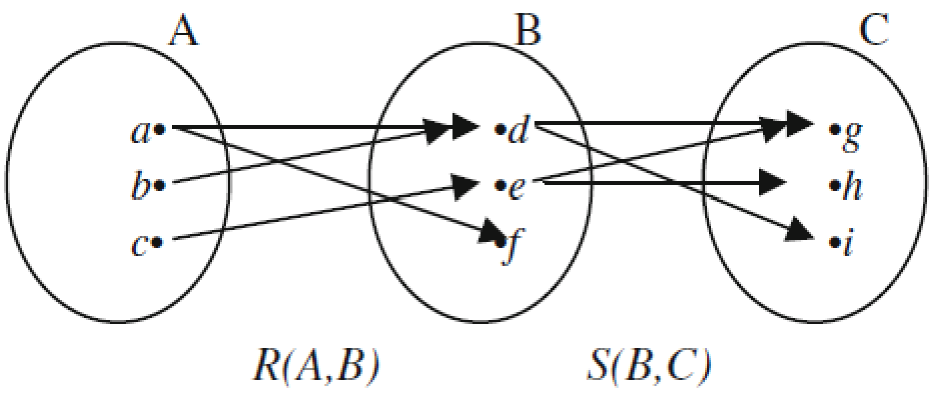
$$(\boldsymbol{R_3\ o\ R_2})\ \boldsymbol{o}\ \boldsymbol{R_1} = \boldsymbol{R_3\ o\ (R_2\ o\ R_1)}$$

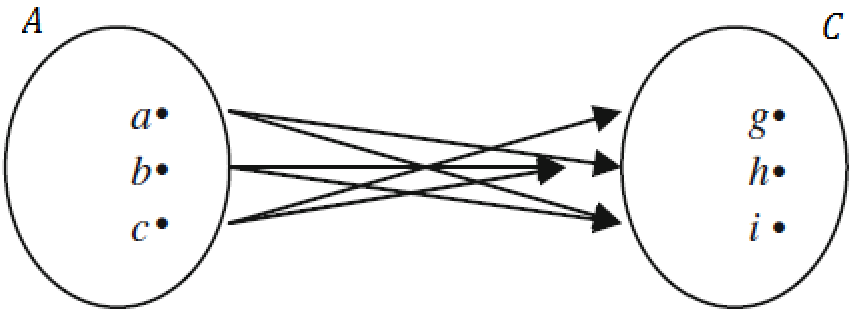
Proof. Exercise.

Example 3.2.19.

(i) Let sets $A = \{a, b, c\}, B = \{d, e, f\}, C = \{g, h, i\}$ and relations $R(A,B) = \{(a,d), (a,f), (b,d), (c,e)\}$ and $S(B,C) = \{(d,g), (d,i), (e,g), (e,h)\}$. Then we graph these relations and show how to determine the composition pictorially S o R is determined by choosing $x \in A$ and $y \in C$ and checking if there is a route from x to y in the graph. If so, we join x to y in $S\ o\ R$.

$$S\ o\ R = \{(a,g), (a,i), (b,g), (b,i), (c,g), (c,h)\}.$$





$S \circ R$

For example, if we consider a and g we see that there is a path from a to d and from d to g and therefore (a, g) is in the composition of S and R.
(ii) Let $R^{-1} = \{(b, a)|(a, b) \in R\}$. The composition of R and R^{-1} yields:

$$R^{-1} \text{ o } R = \{(a, a)|a \in \text{Dom}(R)\} = I_A$$

and

$$R \text{ } o \text{ } R^{-1} = \{(b, b)|b \in \text{Dom}(R^{-1})\} = I_B.$$

Definition 3.2.19. Union and Intersection of Relations
(i) The union of two relations $R_1(A, B)$ and $R_2\,(A, B)$ is subset of $A \times B$ and defined as

$$(a, b) \in R_1 \cup R_2 \text{ if and only if } (a, b) \in R_1 \text{ or } (a, b) \in R_2.$$

(ii) The intersection of two relations $R_1(A, B)$ and $R_2\,(A, B)$ is subset of $A \times B$ and defined as

$$(a, b) \in R_1 \cap R_2 \text{ if and only if } (a, b) \in R_1 \text{ and } (a, b) \in R_2.$$

Remark 3.2.20.
(i) The relation R_1 is a subset of R_2 $(R_1 \subseteq R_2)$ if whenever $(a, b) \in R_1$ then $(a, b) \in R_2$.
(ii) The intersection of two equivalence relations R_2, R_1 on a set X is also equivalence relation on X.
(iii) In general, the union of two equivalence relations R_2, R_1 on a set X need not to be an equivalence relation on X.

Proof. Exercise.