CHAPTER THREE: The Equivalent Barotropic Model

3.1 Introduction

The equivalent barotropic model was developed by ***Charney*** and ***Eliassen*** in 1949. It is a very simple model whose horizontal motions can be identified with those of the atmosphere at particular level.

In this model the wind maintains a constant direction with height. However it's sped varies in a similar way along each vertical. The model is sometimes referred to as the barotropic model, because it become equivalent to it when applied at a so called level of non –divergences.

The assumptions concerning the variation of the wind velocity in the vertical imply that the isotherms in an isobaric surface coincide exactly with its contours. The fat that the model is still widely used today indicates that in mid-troposphere the equivalent barotropic model is a reasonable representative of atmospheric conditions except in periods of baroclinic instability.

3.2 The Governing Equation

One of the early models which played a major role in practical numerical prediction is the barotropic model. It is usually applied at the 500 mb level where it is assumed that the atmosphere moves horizontally without divergence. Under these conditions the vorticitye equation is applied in the simplified form for non-divergence:

$$\frac{d(\rho + f)}{dt} = 0 \quad \text{---------------- --------------------------} \quad (3.1)$$

Expanding

$$\frac{\partial(\rho + f)}{\partial t} + \vec{V}.\nabla_p(\rho + f) + w\frac{\partial(\rho + f)}{\partial p} = 0$$

For horizontal motion $(w = 0)$ the governing equation becomes

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla(\rho + f) = 0 \quad \text{------------ ----------------------} \quad (3.2)$$

Where the subscript for the isobaric surface has been omitted.

It is not immediately evident that equation (3.2) is applicable to the real atmosphere at the 500 mb level. In order to justify this approximation we shall study an equivalent barotropic model.

3.3 The equivalent Barotropic Model

The equivalent barotropic model is defined as a model atmosphere in which the vertical variation of the (geostrophic) horizontal wind is given by the relation.

$$\vec{V}(x, y, p, t) = A(p)\vec{V}(x, y, t) \quad \text{------ -----------------} \quad (3.3)$$

Where:

1. $A(p)$ is an empirical function of pressure describing the vertical vertically of the wind speed.

2. $\overline{\vec{V}}$ vertical averaged wind

i.e. $$\overline{\vec{V}} = \frac{1}{p_o}\int_0^{p_o} \vec{V}\,dp \quad \text{-------------- ------------------------------ (3.4)}$$

Where p_o pressure at lower boundary of the atmosphere.

Note:

1. From equation (3.3) and (3.4) it is evident that

$$\overline{A(p)} = 1 \quad \text{------------ ----------------------------------- (3.5)}$$

i.e. the vertical average of $A(p) = 1$

2. The implication of $A(p)$ is that while the wind may change speed with height, it will always have the same direction, i.e. the direction of $\overline{\vec{V}}$.
3. the thermal wind is

$$\frac{\partial \vec{V}}{\partial p} = \frac{dA}{dp}\overline{\vec{V}} \quad \text{----------- ------------------------------- (3.6)}$$

Now the thermal wind has some direction as $\overline{\vec{V}}$, but it is also directed along the isothermals. Hence, it follows that in this model the isotherms at any level are parallel to the contours. i.e. there is no temperature advection.

3.4 Vorticity Equation for the Vertical Mean Flow

Consider to the vorticity equation:

$$\frac{\partial \rho}{\partial t} + \vec{V}.\nabla(\rho + f) = f\frac{\partial w}{\partial p} \quad \text{------------ --------------- (3.7)}$$

Where we have used the equation of continuity for the p-system

$$\nabla.\vec{V} = -\frac{\partial w}{\partial p} \quad \text{-------------------------- --------------- (3.8)}$$

From equation (3.3) we have

$$\rho = A(p)\overline{\rho} \quad \text{------------------------- ----------------- (3.9)}$$

Integrating (3.7) through the depth of the atmosphere and taking the mean value

$$\overline{\frac{\partial \rho}{\partial t}} + \overline{\vec{V}.\nabla(\rho + f)} = f\,\overline{\frac{\partial w}{\partial p}} \quad \text{--------- ----------------- (3.10)}$$

Using the simplified boundary conditions w=0 at p=0 and $p_o = 0$

$$\frac{\partial \overline{A\rho}}{\partial t} + \overline{A\vec{V}.\nabla\left(A\bar{\rho} + f\right)} = 0$$

$$\bar{A}\frac{\partial \bar{\rho}}{\partial t} + \overline{A^2}\vec{V}.\nabla\bar{\rho} + \bar{A}\vec{V}.\nabla f = 0 \qquad \text{-------------------- (3.11)}$$

$$\frac{\partial \bar{\rho}}{\partial t} + \overline{A^2}\vec{V}.\nabla\bar{\rho} + \vec{V}.\nabla f = 0$$

This is the vorticity equation for the horizontal flow averaged through the vertical mean flow.

3.5 The Equivalent Barotropic Level

The shape of the curve for the function $A(p)$ is show by the full curve with its vertical average of 1.

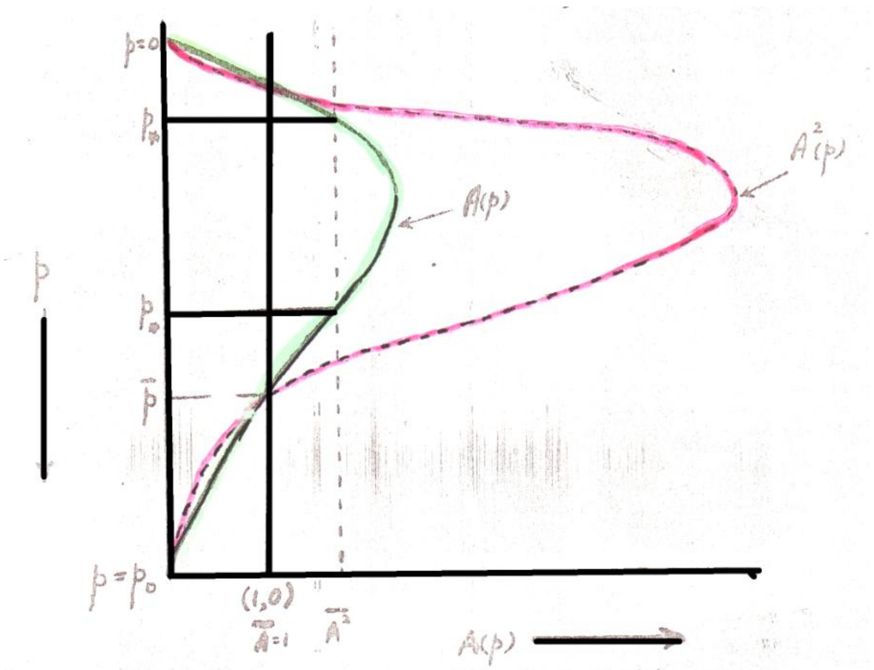

Figure 3.1 Graph of A(p)

The dash curve is the function A^2 shoes vertical averaged $\overline{A^2}$ occurs in (3.11).

We now define an equivalent barotropic level $p = p_o$ such that:

$$A_* = A(p_*) = \overline{A^2} \qquad \text{------------------- ---------------------- (3.12)}$$

Hence

$$\rho_* = A_*\bar{\rho}$$
$$\vec{V}_* = A_*\vec{V} \qquad \text{------------------ ----------------------------- (3.13)}$$

Substituting (3.12) and (3.13) in (3.11)

$$\frac{1}{A_*}\frac{\partial \rho_*}{\partial t} + \frac{\overline{A^2}}{A_*}\vec{V}_*.\nabla \frac{1}{A_*}\rho_* + \frac{1}{A_*}\vec{V}_*.\nabla f = 0$$

OR

$$\frac{\partial \rho_*}{\partial t} + \vec{V}_*.\nabla(\rho_* + f) = 0 \qquad \text{--------------- --------------- (3.14)}$$

Comparing this equation with (3.2) shows that the equivalent barotropic model atmospheric model atmosphere behaves as if it were barotropic at the level where $p = p_o$. Hence $p = p_o$ is called the *equivalent barotropic level*.

Figure 3.1 show that the level $p = p_o$ is located somewhat higher in the atmosphere than the level $p = \bar{p}$ where the wind is equal to the mean wind $\vec{V}$.

In practice it is found that $\bar{p}$ is located around 600 mb, while p_* is located at 500 mb approximately. However, the position of the equivalent barotropic level (p_*) is really a function of location and time. In numerical weather prediction it is assumed that p_* is located at constant pressure (500 mb) and this discrepancy may explain some of the errors in the use of the equivalent barotropic forecasts.

3.6 Vertical Velocities

The equation (3.14) only applies at the level $p = p_o$. At other levels there will be vertical velocities.

$$\frac{1}{A_*}\frac{\partial \rho_*}{\partial t} + A^2\vec{V}.\nabla\bar{\rho} + A\vec{V}.\nabla f = f\frac{\partial w}{\partial p} \qquad \text{------------------- -------- (3.15)}$$

Multiply (3.11) by $A(p)$ and subtract the result from (3.15)

$$A^2\vec{V}.\nabla\bar{\rho} - A\overline{A^2}\vec{V}.\nabla\bar{\rho} = f\frac{\partial w}{\partial p}$$

OR

$$\frac{\partial w}{\partial p} = \left(A^2 - \overline{A^2}A\right)\frac{1}{f}\vec{V}.\nabla\bar{\rho} \qquad \text{---------------------------- ----------------- (3.16)}$$

i.e.

$$w = c(p)\frac{1}{f}\vec{V}.\nabla\bar{\rho} \qquad \text{--------------------------------- ----------------- (3.17)}$$

Where

$$c(p) = \int_0^p \left(A^2 - \overline{A^2}A\right)dp \qquad \text{--------------------------- ------------------ (3.18)}$$

Note that $c(p) = 0$, such that lower boundary condition $(w = 0 .... at .. p = p_o)$ is satisfied.

Equation (3.17) indicates the w is proportional to the vorticity advection $\vec{V}.\nabla\bar{\rho}$.

3.7 The Interpretation of $c(p)$

Introduce a local coordinate system in which the s-axis in along the streamline and the n-axis is normal to the streamline.

$$\vec{V}.\nabla\bar{\rho} = \bar{V}\frac{\partial\bar{\rho}}{\partial s} \quad \text{--------------- ----------------------------------} \quad (3.19)$$

Consider (3.17) and (3.19) in the case in which $c(p)\ \phi\ 0$. It can be seen that w is positive (i.e. subsidence occurs) when the wind blows from low to high values of the vorticity, i.e. $\frac{\partial\bar{\rho}}{\partial s}\ \phi\ 0$. By contrast, ascent occurs if w is negative, i.e. if $\frac{\partial\bar{\rho}}{\partial s}\ \pi\ 0$.

As a special result, we have subsidence between the ridge and the trough looking downstream. On the other hand, rising motion occurs between the trough and the ridge.

The above discussion applies in the case in which $c(p)\ \phi\ 0$. The opposite conditions apply if $c(p)\ \pi\ 0$. In order to understand the effect of $c(p)$, consider a simple example. Let

$$A = A(p) = \begin{cases} 2\dfrac{p}{p_T}\text{........ } if \text{ .... } 0 \le p \le p_T \text{ ..... } where \text{ ..... } p_T = 250\ mb \\ 2\dfrac{p_o - p}{p_o - p_T}\text{........ } if \text{ .... } p_T \le p \le p_o \text{..... } where \text{ ..... } p_o = 1000\ mb \end{cases}$$

It can be seen that $\overline{A^2} = \frac{4}{3}$

$$\therefore \qquad c(p) = \begin{cases} -\dfrac{4}{3}\dfrac{p}{p_T}(p_T - p)\text{....... } if \text{ ...... } 0 \le p \le p_T \\ -\dfrac{4}{3}\dfrac{(p_o - p)^2}{(p_o - p_T)^2}(p - p_T)\text{........ } if \text{ .... } p_T \le p \le p_o \end{cases}$$

The function $A(p)$ in equation (3.11) is a simple liner function. It is realistic in the sense that it has a maximum at 250 mb which simulates the maximum (jet-stream) in a real atmosphere.

Note that the troposphere level where the actual wind $\vec{V} = \vec{V}$ (i.e. A=1) is found at $\bar{p} = 625\ mb$. The tropospheric equivalent barotropic is at $p_{*}500\ mb$.

Now $c\ \pi\ 0 \text{.... } if \text{ .... } p\ \pi\ p_T$ (the stratosphere)

$c\ \phi\ 0 \text{....... } if \text{ .... } p_T\ \pi\ p_o$

Zero values of c occur at $p = 0,\text{.... } p = p_T \text{ ... } and \text{ ... } p = p_o$

Note that the Non-divergent levels occur where $\frac{\partial w}{\partial p} = 0$ from equation (3.16) these occur where:

$$A^2 - \overline{A^2}A = 0 \quad \text{------------------ ------------------------} \quad (3.20)$$

i.e.

$$A = \overline{A^2} \qquad \text{------------------------ -----------------------} \qquad (3.21)$$

Hence the equivalent barotropic level $p = p_*$ is also a level of non-divergence as one would expect from (3.14).

Figure 3.2 show the functions $A(p) \ldots \text{ and } \ldots c(p)$ for the example discussed above.

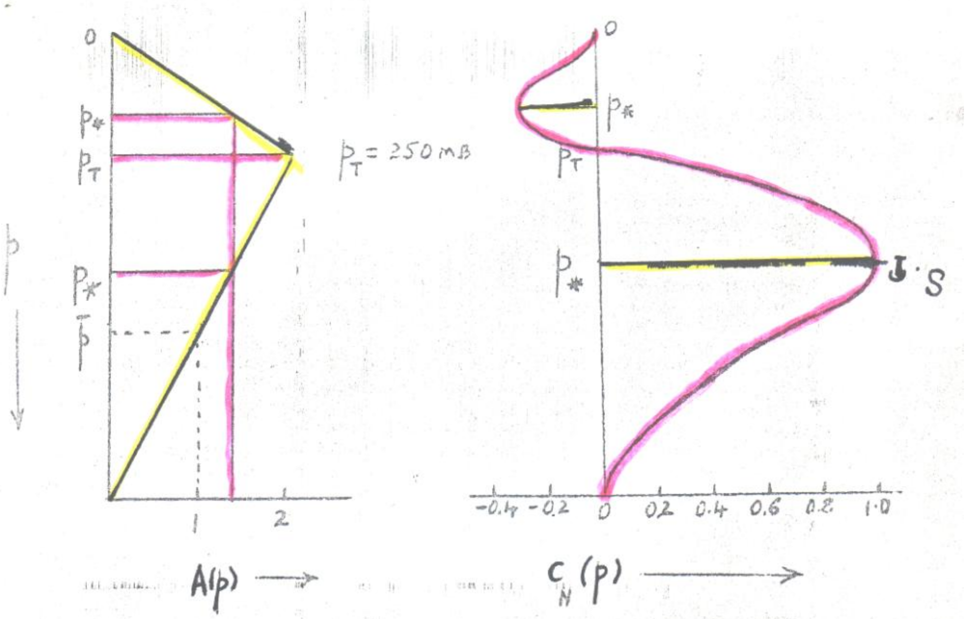

Figure 3.2 Special example of $A(p) \ldots \text{ and } \ldots c(p)$.

The profile of $c(p)$ has been normalized so that its tropospheric maximum is unity. It can be seen that $c_N(p)$ has opposite signs below and above the wind maximum $(at \ldots p = p_T)$.

The existence of the maximum of in the middle of the atmosphere gives convergence above and divergence below if the vorticity advection $\overline{V}\dfrac{\partial \overline{\rho}}{\partial s}$ is positive. This leads to subsidence $(w \phi 0)$.

The opposite situation occur (i.e. ascent) if the vorticity advection is negative.

3.8 Application of the Equivalent Barotropic Model

The properties of the equivalent barotropic model are the main reasons for the application of the barotropic vorticity equation (3.2) at the level 500 mb. Such barotropic forecasts are made routinely by all major numerical forecast centers in the world for periods up to 72 hours.

The very fact that barotropic forecasts are made many years after the model was first introduced is significant. It means that in spite of its simplicity, the model describes a major part of the changes in the large-scale atmospheric flow.