ESCI 485 – Air/sea Interaction
Lesson 4 – Waves
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GENERAL WAVE PROPERTIES

- **Each type of wave has its own dispersion characteristics, or *dispersion relation*.**
  - **The dispersion relation is an equation that gives the frequency (ω) of the wave as a function of physical parameters and/or the wave number.**
- **The speed of an individual wave crest is called the *phase speed*, and is given as**

$$c_p \equiv \omega / k$$

- **The energy of the wave propagates at the *group velocity*. The group velocity is not necessarily the same as the phase speed. The group velocity is found by**

$$c_g \equiv \frac{\partial \omega}{\partial k}.$$

- **If the phase speed of a wave doesn't depend on wave number then the wave is called *non-dispersive*. In this case it turns out that $c_g = c_p$.**
- **If the phase speed is a function of wave number then the waves are *dispersive*.**

DISPERSION CHARACTERISTICS OF CAPILLARY WAVES

- **The dispersion relation for surface capillary/gravity waves is**

$$\omega = \pm\sqrt{\left(kg + k^3\,\sigma/\rho\right)\tanh kH}$$

where σ is the surface tension.

- **The surface tension term is only important for the shortest wavelengths (on the order of a few centimeters or less).**
  - **For wavelengths smaller than around 1.7 cm only surface tension is important and the waves are pure capillary waves.**
- **The dispersion relation for pure capillary waves is**

$$\omega = \pm\sqrt{k^3\left(\sigma/\rho\right)\tanh kH}\ .$$

- **Capillary waves are anomalously dispersive, meaning that the group velocity is larger than the phase speed and that shorter wavelengths are faster than longer wavelengths.**
- **Capillary waves attenuate very quickly due to molecular viscosity. They are not found far from the source region.**

DISPERSION CHARACTERISTICS OF SURFACE GRAVITY WAVES

- **For wavelengths longer than a few centimeters the surface tension can be ignored. We therefore have the dispersion relation for pure gravity waves,**

$$\omega = \pm\sqrt{kg \tanh kH}\ .$$

- **We can break these surface gravity waves into three 'regimes'**
- **Deep water waves**
  - **These are waves whose wavelength is much less than the depth of the water. In this case *kH* is large, and so the dispersion relation becomes**

$$\omega = \pm\sqrt{kg}\ .$$

  - **Deep water waves are dispersive**
  - **The pressure perturbation in deep water waves decreases exponentially with depth, with an *e*-folding scale on the order of the wavelength.**
  - **Deep water waves are non-hydrostatic.**
- **Shallow water waves**
  - **These are waves whose wavelength is much greater than the depth of the water. In this case, *kH* is small, so that**

$$\tan kH \cong kH\ ,$$

    and the dispersion relation becomes

$$\omega = \pm k\sqrt{gH}\ .$$

  - **Shallow water waves are non-dispersive.**
  - **The shallower the water, the slower the wave.**
  - **Shallow-water waves are hydrostatic.**
  - **The pressure perturbation has the same magnitude all the way to the bottom of the fluid.**
- **In between the deep water and shallow water waves we must use the full dispersion relation for gravity waves.**

SEA AND SWELL

- **The waves on the ocean are broken into two categories.**
- ***Sea* – steep, irregular waves.**

- Sea dissipates quickly after the wind dies

- *Swell* – **regular, longer, low, and rounded waves that are left after the wind dies down, or that propagate away from the windy region.**
  - **Swell can propagate for thousands of miles.**

INITIAL GENERATION OF WIND WAVES

- **The surface waves observed on the ocean are generated by the wind.**
- **In a laboratory, when air is blown over initially calm water**
  - **a thin shear flow develops in the upper surface of the water**
  - **turbulence develops in the air directly above the water.**
  - **Long-crested, regular waves quickly develop on the surface.**
  - **After 10 seconds or so, the long-crested, regular waves begin change into short-crested instability waves.**
  - **The instability waves appear simultaneously with turbulence in the shear flow on the water side.**
- **The instability waves have wavelengths near 1.7 cm, which is also the wavelength of minimum speed for surface gravity/capillary waves.**
- **In nature this phenomenon occurs with wind gusts, generating the familiar *cat's paws* on the surface of the water.**

MEASURES OF WAVES

- **Autocorrelation function**

$$R(t) = \overline{\eta(t_0)\eta(t_0 + t)}.$$

  - **The time between peaks in the autocorrelation function is an indicator of the dominant period of the waves.**
  - **The series of peaks in the autocorrelation function decay over about 5 periods. This is why waves often seem to come in sets of five or so.**
- **Power spectrum**
  - **The Fourier coefficients for the waves are found by taking the Fourier transform of the ocean height time series data,**

$$F(\omega) = \int_{-\infty}^{\infty} \eta(t) \exp(-i\omega t)\, dt .$$

- **Plotting the magnitude of $F(\omega)$ as a function of ω gives us a plot of the power spectrum of the waves, and can tell us something about which frequencies of waves contain the most energy.**
- **As the wind becomes stronger, the spectral peak becomes taller and moves to lower frequencies.**
- **The *spectral density***

  $$\phi(\omega)$$

  is just the power spectrum divided by the frequency range (called *bandwidth*) over which each Fourier coefficient is valid.
  - **Just remember that spectral density is proportional to the power spectrum.**
- **Wave spectra usually exhibit a single narrow peak which corresponds to the dominant frequency of the waves.**

- **Root mean square elevation**

$$H_{rms} = \left(\overline{\eta^2}\right)^{1/2} ,$$

- **Significant wave height, $H_{1/3}$ or H_s, defined as the mean of the 1/3 highest waves.**
  - **In the absence of swell the following relation holds**

$$H_s = 4.0 H_{rms} .$$

WAVES IN A FULLY DEVELOPED SEA

- **A *fully developed sea* is one in which the waves have grown to their maximum amplitude for the given wind conditions. This implies**
  - **The wind has been blowing for a long enough duration so that the wave spectrum has become saturated (no more energy can be added).**
  - **The sea is not *fetch-limited*, meaning that the shore is far away.**
- **In a fully developed sea the factors that we would expect to be important for describing the wave field are the spectral density, the wind speed at ten-meters above the surface, wave frequency, and gravity. These can be formed into two non-dimensional groups,**

$$\frac{\phi(\omega) g^3}{U_{10}^5} \quad and \quad \frac{\omega_p U_{10}}{g}.$$

(Note: u^* can be used instead of U_{10}. A typical relation between these two values is $U_{10}/u^* = 27.5$.)

- **We expect these groups to be related, and they have been found to relate quite well.**
- **If these non-dimensional groups are plotted against each other the graphs are nearly identical regardless of the wind speed.**

- **Observations in fully developed seas also show relations between other non-dimensional groups formed from other wave parameters. The following relations have been found to hold**

$$\frac{gH_{rms}}{U_{10}^2} = 0.052$$
$$\frac{gH_s}{U_{10}^2} = 0.2$$

  - **In a fully developed sea, expressions such as these can be used to estimate wave height from the 10-meter wind speed.**

- **The characteristic wave frequency (ω_p) is related to U_{10} and g via**

$$\frac{U_{10}\omega_p}{g} = 0.88,$$

which shows that as wind speed increases, the characteristic wave frequency decreases.

  - **If we assume the waves are deep-water waves, then**

$$\omega_p^{\ 2} = k_p g = \frac{\omega_p}{C_p} g$$

where C_p is the phase speed of the dominant wave, so that

$$\omega_p = g/C_p.$$

Combined with the expression in the preceding bullet, we get a relation between the 10-meter wind and the speed of the characteristic wave,

$$C_p/U_{10} = 1.14.$$

- **The characteristic moves slightly faster than the 10-meter wind, a relationship that can sometimes be casually observed if you sail downwind at the speed of the wind.**

WAVE GROWTH

- **Near the coastline, or in regions where the wind has not acted long enough for the seas to become fully developed, an additional parameter that must be taken into account is the fetch, denoted by *X*.**
- **An additional non-dimensional group that can be formed is gX/u^{*2}, so that there are now three non-dimensional groups**

$$\frac{\phi(\omega)g^3}{u^{*5}}, \quad \frac{\omega_p u^*}{g}, \quad and \quad \frac{gX}{u^{*2}} \quad .$$

- **We expect relationships to hold between these three non-dimensional groups. One such empirical relation that has been found is**

$$\frac{u^* \omega_p}{g} = 7.1\left(\frac{gX}{u^{*2}}\right)^{-1/3},$$

which relates the characteristic wave frequency to the fetch, and shows that as fetch increases the wave frequency decreases.

- **The rms wave height has been found to relate to fetch via**

$$\frac{g^2 H_{rms}^2}{u^{*4}} = \left(1.6\times10^{-4}\right)\frac{gX}{u^{*2}},$$

showing wave height increases with the square-root of the fetch.

- **These two relations can be combined, eliminating the fetch, to get**

$$\frac{gH_{rms}}{u^{*2}} = 0.057\left(\frac{u^* \omega_p}{g}\right)^{-3/2}.$$

Using the relations $H_s = 4H_{rms}$, and the definition of period, $T_p = 2\pi/\omega_p$, we get

$$\frac{gH_s}{u^{*2}} = 0.061\left(\frac{gT_p}{u^*}\right)^{3/2}.$$

This relation shows that as the wave height increases, so does the period of the characteristic wave.

- **Since the longer waves travel faster than the shorter waves, and as the waves grow the characteristic period becomes longer, than as time goes on the phase speed (C_p) of the dominant wave increases. Therefore, the ratio of $C_p/u*$ can be used as a non-dimensional *wave age*.**
- **A measure of *wave steepness* is H_s/λ_p, where λ_p is the wavelength of the characteristic wave.**
- **From the relationships above, a formula for wave steepness as a function of wave age can be found to be**

$$\frac{H_s}{\lambda_p} = 0.153\left(\frac{C_p}{u*}\right)^{-1/2},$$

and shows that the waves become less steep with time.

WIND WAVE FORECASTING

- **Theory and observation has allowed us to come up with several formulas relating important wave parameters (height and period notably) in terms of the 10-meter wind speed, fetch, and duration of the wind.**
  - **The formulas are usually displayed in graphical form.**
- **The key to a good wave-height forecast is to have a good forecast of the wind field.**
- **Fetch must be accounted for even in the middle of the ocean.**
- **If the sea is not fully developed due to inadequate fetch, the waves are *fetch-limited*.**
- **If the sea is not fully developed due to inadequate duration of wind, the waves are *duration-limited*.**
- **The effects of swell must also be taken into account when forecasting waves.**
  - **The combined effects of *sea* and *swell* are accounted for by defining a combined wave height, according to the formula**

$$H_s^{combined} = \sqrt{\left(H_s^{sea}\right)^2 + \left(H_s^{swell}\right)^2}\ .$$