

## *First Order and First Degree*

We will discuss the standard methods of solving the D.E. of the following:

### *1) separation of the variables equation:*

If a differential equation can be written in the form

$$f(y)dy = \varphi(x)dx$$

We say that variables are separable,  $y$  on left hand side and  $x$  on right hand side. We get the solution by integrating both sides.

#### *Working Rule:*

*Step 1. Separate the variables as  $f(y)dy = \varphi(x)dx$*

*Step 2. Integrate both sides as  $\int f(y) dy = \int \varphi(x)dx$*

*Step 3. Add an arbitrary constant  $C$  on R.H.S.*

**Ex. (1):** Solve  $\frac{dy}{dx} = x + xy$  ?

#### **Solution:**

$$\frac{dy}{dx} = x(1 + y) \quad \Rightarrow \quad \frac{dy}{(1 + y)} = x dx \quad \Rightarrow \quad \int \frac{dy}{1 + y} = \int x dx$$

$$\ln(1 + y) = \frac{x^2}{2} + C \quad \Rightarrow \quad 1 + y = \exp\left(\frac{x^2}{2} + C\right)$$

$$1 + y = A \exp\left(\frac{x^2}{2}\right)$$

where  $c$  and hence  $A$  is an arbitrary constant.

**Ex. (2):** Solve  $x^2(1 - y^2) dx + y(1 + x^2) dy = 0$  ?

**Solution:**

$$\frac{x^2 dx}{1 + x^2} + \frac{y dy}{1 - y^2} = 0$$

$$\int \left(1 - \frac{1}{1 + x^2}\right) dx - \frac{1}{2} \int \frac{-2y}{1 - y^2} dy = C$$

$$x - \tan^{-1} x - \frac{1}{2} \ln(1 - y^2) = C$$

**Ex. (3):** Solve  $\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$  ?

**Solution:**

$$(\sin y + y \cos y) dy = \{x(2 \log x + 1)\} dx$$

$$\int (\sin y + y \cos y) dy = \int \{x(2 \log x + 1)\} dx$$

$$-\cos y + y \sin y - \int (1) \sin y dy = 2 \int \log x x dx + \int x dx + C$$

$$-\cos y + y \sin y + \cos y = 2 \left[ \log x \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} dx \right] + \frac{x^2}{2} + C$$

$$y \sin y = 2 \log x \frac{x^2}{2} - \int x dx + \frac{x^2}{2} + C$$

$$y \sin y = 2 \log x \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$y \sin y = x^2 \log x + C$$

**2) Exact equation:**

$$Mdx + Ndy = \theta$$

an exact D.E. if it satisfies the following condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

**Proof:**

$$Mdx + Ndy = dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$M(x, y) = \frac{\partial U}{\partial x} , \quad N(x, y) = \frac{\partial U}{\partial y}$$

Since  $\partial^2 U / \partial x \partial y = \partial^2 U / \partial y \partial x$  we therefore require

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

**Working Rule:**

*Step 1. Integrate M with respect to x keeping y constant.*

*Step 2. Integrate w.r.t. y, only those terms of N which do not contain x.*

*Step 3. Result of (1) + Result of (2) = Constant.*

**Ex. (4):** Solve  $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

**Solution:**

$$M = 5x^4 + 3x^2y^2 - 2xy^3, \quad N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 , \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \int (5x^4 + 3x^2y^2 - 2xy^3)dx + \int -5y^4dy = C$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

**Ex. (5):** Solve  $x \frac{dy}{dx} + 3x + y = 0$  ?

**Solution:**

$$(3x + y)dx + xdy = 0,$$

$$M(x, y) = 3x + y, \quad N(x, y) = x \quad \text{Since} \quad \partial M / \partial y = 1 = \partial N / \partial x$$

$$\int (3x + y)dx + \int (0)dy = C$$

$$\frac{3x^2}{2} + yx = C.$$

**Ex. (6):** Solve  $(3x^2 + 3xy^2) dx + (3x^2y - 3y^2 + 2y) dy = 0$  ?

**Solution:**

$$M(x, y) = 3x^2 + 3xy^2, \quad N(x, y) = 3x^2y - 3y^2 + 2y$$

$$\frac{\partial M}{\partial y} = 6xy, \quad \frac{\partial N}{\partial x} = 6xy$$

$$\therefore \int (3x^2 + 3xy^2)dx + \int (-3y^2 + 2y)dy = C$$

$$x^3 + \frac{3}{2}x^2y^2 - y^3 + y^2 = C.$$

**Ex. (7):** Solve  $[1 + \log(xy)]dx + \left[1 + \frac{x}{y}\right]dy = 0$  ?

**Solution:**

$$[1 + \log x + \log y]dx + \left[1 + \frac{x}{y}\right]dy = 0$$

$$M(x, y) = 1 + \log x + \log y, \quad N(x, y) = 1 + \frac{x}{y}$$

$$\frac{\partial M}{\partial y} = \frac{1}{y}, \quad \frac{\partial N}{\partial x} = \frac{1}{y}$$

$$\therefore \int (1 + \log x + \log y)dx + \int dy = C$$

$$x + \int \log x dx + \int \log y dx + y = C \quad \dots \dots (*)$$

Now,

$$\int \log x dx = x \log x - \int \frac{1}{x} \cdot x dx = x \log x - x$$

Equation (\*) becomes

$$x + x \log x - x + x \log y + y = C$$

$$x[\log x + \log y] + y = C = x \log(xy) + y = C.$$

**3) Inexact equation: Integration Factors:**

$$Mdx + Ndy = \theta$$

an inexact D.E. if it satisfies the following condition

$$IF \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \quad \text{we must use Integration Factor } (\mu)$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}, \text{ there exists no general method for finding it.}$$

**Working Rule:**

$$\text{Step 1. } \mu(x) = \exp \int f(x) dx, \quad f(x) = \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right).$$

$$\text{Step 2. } \mu(y) = \exp \int g(y) dy, \quad g(y) = \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right).$$

Step 3. Multiple D.E. by step (1) **OR** step (2).

Step 4. Apply exact method.

**Ex. (8):** Solve  $\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}$  ?

**Solution:**

$$(4x + 3y^2)dx + 2xydy = 0$$

$$M(x, y) = 4x + 3y^2 \quad \text{and} \quad N(x, y) = 2xy$$

$$\frac{\partial M}{\partial y} = 6y \neq \frac{\partial N}{\partial x} = 2y$$

$$f(x) = \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{2}{x}$$

$$\therefore \mu(x) = \exp 2 \int \frac{dx}{x} = \exp (2 \ln x) = x^2$$

$$(4x^3 + 3x^2y^2)dx + 2x^3ydy = 0$$

$$\int (4x^3 + 3x^2y^2)dx + \int (0)dy = C$$

$$x^4 + y^2x^3 = C.$$

**Ex. (9):** Solve  $(3xy^3 + 4y) dx + (3x^2y^2 + 2x) dy = 0$  ?

**Solution:**

$$M(x, y) = 3xy^3 + 4y, \quad N(x, y) = 3x^2y^2 + 2x$$

$$\frac{\partial M}{\partial y} = 9xy^2 + 4 \neq \frac{\partial N}{\partial x} = 6xy^2 + 2$$

$$f(x) = \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{3x^2y^2 + 2x} ([9xy^2 + 4] - [6xy^2 + 2]) = \frac{1}{x}$$

$$\therefore \mu(x) = \exp \int \frac{dx}{x} = \exp(\ln x) = x$$

$$(3x^2y^3 + 4xy)dx + (3x^3y^2 + 2x^2)dy$$

$$\therefore \int (3x^2y^3 + 4xy)dx + \int (0)dy = C$$

$$x^3y^3 + 2x^2y = C.$$

**Ex. (10):** Solve  $(2x \log x - xy)dy + 2ydx = 0$  ?

**Solution:**

$$M(x, y) = 2y, \quad N(x, y) = 2x \log x - xy$$

$$\frac{\partial M}{\partial y} = 2 \neq \frac{\partial N}{\partial x} = 2(1 + \log x) - y$$

$$\begin{aligned} f(x) &= \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{2x \log x - xy} ([2] - [2(1 + \log x) - y]) \\ &= -\frac{1}{x} \end{aligned}$$

$$\therefore \mu(x) = \exp \int -\frac{dx}{x} = \exp(-\ln x) = \frac{1}{x}$$

$$(2 \log x - y)dy + \frac{2y}{x}dx = 0$$

$$\int \frac{2y}{x}dx + \int -ydy = C$$

$$2y \log x - \frac{y^2}{2} = C.$$

**Ex. (11):** Solve  $(2xy^2 - 2y) dx + (3x^2y - 4x) dy = 0$  ?

**Solution:**

$$M(x, y) = 2xy^2 - 2y, N(x, y) = 3x^2y - 4x$$

$$\frac{\partial M}{\partial y} = 4xy^2 - 2 \neq \frac{\partial N}{\partial x} = 6xy - 4$$

$$g(y) = \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{2xy^2 - 2y} ([6xy - 4] - [4xy - 2]) = \frac{1}{y}$$

$$\therefore \mu(y) = \exp \int \frac{dy}{y} = \exp (\ln y) = y$$

$$(2xy^3 - 2y^2) dx + (3x^2y^2 - 4xy) dy = 0$$

$$\therefore \int (2xy^3 - 2y^2)dx + \int (0)dy = C$$

$$x^2y^2 - 2xy^2 = C.$$

**Ex. (12):** Solve  $(y^4 + 2y) dx + (xy^3 + 2y^2 - 4x) dy = 0$  ?

**Solution:**

$$M(x, y) = y^4 + 2y, N(x, y) = xy^3 + 2y^2 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \neq \frac{\partial N}{\partial x} = y^3 - 4$$