Proof. The mapping $\varphi'([s]_\rho) = \varphi(s)$ for any $s \in S$ makes the diagram commutative by the Homomorphism Theorem 1.1.8 for sets and is obviously a homomorphism of semigroups. □

Corollary 2.27. *If $\varphi : S \to T$ is a surjective homomorphism of semigroups then $T \cong S/\ker\varphi$.*

Exercise 2.28.

(1) Let $\varphi : S \to T$ and $\psi : T \to U$ be semigroup isomorphisms. Then the product $\psi\varphi : S \to U$ is also an isomorphism.

Let $\varphi : S \to T$ be a semigroup isomorphism. Then the inverse mapping $\varphi^{-1} : T \to S$ is also an isomorphism.

Thus, the relation $S \cong T$ for semigroups is an equivalence relation on the class of semigroups.

(2) Let $\varphi : S \to G$ be a semigroup isomorphism and G a monoid (group). Then S is a monoid (group).

Free semigroups

Definition 2.29. A subset M of generating elements of a semigroup S is called a ***basis*** of S if every element of S can be uniquely presented as a product of elements of M.

Definition 2.30. A semigroup is called ***free*** if it contains a basis. A monoid T is called ***free*** if $T = S^1$ for a free semigroup S.

Note that the semigroups X^+ and monoids X^* we constructed in Example 1.2.15(1) are free in the sense of Definition 1.2.30.

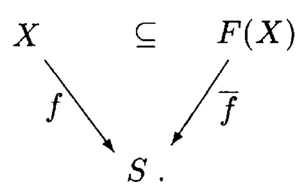
The next proposition justifies the name of free semigroups.

Proposition 2.31. *Let $F(X)$ be a free semigroup with a basis X and $f : X \to S$ any mapping from the basis X into a semigroup S. Then there exists a unique homomorphism $\overline{f} : F(X) \to S$ which extends f, i.e. $\overline{f}|_X = f$.*

Proof. Define the mapping $\overline{f} : F(X) \to S$ by

$$\overline{f}(x_1x_2\dots x_k) = f(x_1)f(x_2)\dots f(x_k)$$

for any $x_1x_2\dots x_k \in F(X)$. Then we have



Since every element of $F(X)$ can be uniquely presented as a product of elements of X, $\overline{f}$ is well and uniquely defined. Now $\overline{f}$ is a homomorphism since for any $x_1x_2\dots x_k,\ y_1y_2\dots y_l \in F(X)$ one has

$$\begin{aligned}\overline{f}((x_1x_2\dots x_k)(y_1y_2\dots y_l)) &= \overline{f}(x_1x_2\dots x_ky_1y_2\dots y_l)\\ &= f(x_1)f(x_2)\dots f(x_k)f(y_1)f(y_2)\dots f(y_l)\\ &= (f(x_1)f(x_2)\dots f(x_k))(f(y_1)f(y_2)\dots f(y_l))\\ &= \overline{f}(x_1x_2\dots x_k)\overline{f}(y_1y_2\dots y_l)\,.\end{aligned}$$

Finally, $\overline{f}(x) = f(x)$ for all $x \in X$ by the definition of $\overline{f}$. □

Corollary 2.32. *Any free semigroup $F(X)$ with a basis X is isomorphic to the free word semigroup X^+.* □

Theorem 2.33. *Every semigroup is isomorphic to a factor semigroup of a free semigroup.*

Proof. Let S be a semigroup. Consider the free semigroup S^+ and the mapping $\mathrm{id}_S : S \to S$. By applying Proposition 1.2.31 we get a homomorphism $g = \overline{\mathrm{id}_S} : S^+ \to S$ such that $g|_S = \mathrm{id}_S$. Clearly, g is surjective. Now we have by Corollary 1.2.27 that $S \cong S^+/\ker g$. □

Cayley's Theorem for semigroups

The following theorem which was first proved by Suschkevitsch [Sus26] is an analog to Cayley's Theorem for groups.

Theorem 2.34. *Every semigroup is isomorphic to a subsemigroup of the semigroup of all transformations on a set.*

Proof. Let S be a semigroup. For every $s \in S$ consider the left translation

$$\lambda_s : S^1 \to S^1$$

defined by

$$\lambda_s(x) = sx$$

for $x \in S^1$. Clearly, $\lambda_s \in \mathcal{T}(S^1)$. Now set $T = \{\,\lambda_s \mid s \in S\,\}$. Suppose $\lambda_s, \lambda_t \in T$. Then

$$(\lambda_s\lambda_t)(x) = \lambda_s(\lambda_t(x)) = \lambda_s(tx) = s(tx) = (st)x = \lambda_{st}(x)$$

for every $x \in S$. This means that $\lambda_s\lambda_t = \lambda_{st}$ and thus $\lambda_s\lambda_t \in T$. Hence T is a subsemigroup of $\mathcal{T}(S^1)$.

Now define a mapping $f : S \to T$ by

$$f(s) = \lambda_s$$

for $s \in S$. Then f is a homomorphism. Indeed,

$$f(st) = \lambda_{st} = \lambda_s \lambda_t = f(s)f(t)$$

for any $s, t \in S$.

If $f(s) = f(t)$, $s, t \in S$, then $\lambda_s = \lambda_t$ and

$$s = s1 = \lambda_s(1) = \lambda_t(1) = t1 = t.$$

Hence f is injective. Moreover, $f : S \to T$ is surjective by the definition of T. Hence $S \cong T$. □

Remark 2.35. Note that the idea behind the proof of Cayley's Theorem for finite semigroups is the representation of a semigroup S as a so-called *Cayley color graph*. Take a set of generating elements C for $S \setminus \{1\}$ and consider the directed colored multigraph $G = (V, E, C)$ with $V = S$ and $(s, s', c) \in E$ if $c \in C$ is such that $sc = s'$. Then c can be viewed as the color of the edge (s, s', c) and it turns out that the monoid of color preserving graph endomorphisms of G is isomorphic to S (cf. Definition 1.1.9).

Comments

This section contains some basic definitions for semigroups and monoids arranged as conveniently as possible for the needs of this book. We gave many examples and "non-examples" and, of course, turned the reader's attention to such basic things as the Homomorphism Theorem for semigroups in a general form based on the corresponding theorem for sets in Section 1.1.

3 Some classes of semigroups

Commutativity, center, centralizer (1.3.1)
Idempotents, bands, semilattices (1.3.2–1.3.7)
Structure of monogenic semigroups (1.3.8–1.3.12)
Bicyclic monoid (1.3.13)
Periodic semigroups, nil semigroups (1.3.14–1.3.17)
Reversible, simple, cancellative, solvable (1.3.18–1.3.22)
Principal ideals and Green's relations (1.3.23–1.2.26)
Regular semigroups (1.3.27–1.3.31)
Orthodox semigroups (1.3.32–1.3.33)
Inverse semigroups (1.3.34–1.3.36)
Right (left) inverse semigroups (1.3.37–1.3.39)

Commutativity, center, centralizer

Definition 3.1. Let S be a semigroup. As in any algebraic structure, we say that $s, t \in S$ ***commute*** if $st = ts$. The set

$$C(S) = \{ c \in S \mid cs = sc \text{ for all } s \in S \}$$

is called the ***center*** of S. If $C(S) = S$ then S is called a ***commutative*** or ***abelian semigroup***. Let $R \subseteq S$ be a subset of the semigroup S. Then $C_S(R) = \{ s \in S \mid rs = sr \text{ for all } r \in R \}$ is called the ***centralizer*** of R in S.

Note that $C(S) = C_S(S)$.

Idempotents, bands, semilattices

Definition 3.2. Let S be a semigroup. An element $e \in S$ is called an ***idempotent*** if $e^2 = e$. The set of all idempotents of S is denoted by $E(S)$. For a subset T of S we write $E(T) = E(S) \cap T$. We define the ***natural partial ordering*** on $E(S)$ by $e \leq f$ if and only if $ef = e = fe$.

There is a wide variation in the number of idempotents a semigroup may contain. For example, $(\mathbb{N}, \cdot)$ has the only idempotent 1, the semigroup $(2\mathbb{N}, \cdot)$ does not contain any idempotents, but in $(\mathbb{N}, \gcd)$ all elements are idempotents.

Definition 3.3. If $E(S) = S$, then S is called an ***idempotent semigroup*** or a ***band***.

Definition 3.4. Let A and B be non-empty sets and set $S = A \times B$. Define a multiplication on S by

$$(a, b)(c, d) = (a, d)$$

for $a, c \in A$, $b, d \in B$. This semigroup is called a ***rectangular band***.

Note that S indeed is a band, which is non-commutative, if either $|A| > 1$ or $|B| > 1$. Moreover, S is a special case of a Rees matrix semigroup without zero (cf. Example 1.2.15 (1)(j)).

The next exercise shows that we may identify the classes of semilattices and commutative bands.

Exercise 3.5. Recall that lower semilattices were defined in Subsection "Order relations" of Section 1.1 as posets in which every two elements have a greatest lower bound.

A commutative band S with the natural order

$$x \leq y \Longleftrightarrow x = xy$$

for $x, y \in S$ is a lower semilattice.

Conversely, if in a lower semilattice P we define a multiplication by

$$xy = x \wedge y$$

for $x, y \in P$ then we get a commutative band.

Exercise 3.6. Let S be a semigroup and $e \in E(S)$. Then

$$H_e = \{ a \in S \mid ae = a = ea \text{ and there exists } a' \in S \text{ such that } aa' = e = a'a \}$$

is the greatest subgroup of S with the identity element e.

Definition 3.7. Let S be a semigroup. For $e \in E(S)$ we call

$$H_e = \{ a \in S \mid ae = a = ea \text{ and there exists } a' \in S \text{ such that } aa' = e = a'a \}$$

the ***group of the idempotent*** e in S.

Structure of monogenic semigroups

Recall that a semigroup S is called monogenic if there exists a subset of generating elements of S consisting of a single element, which is called a generating element of S (see Definition 1.2.19). Now we shall determine the general structure of monogenic semigroups.

Definition 3.8. Let $S = \langle a \rangle = \{ a, a^2, a^3, \dots \}$ be a monogenic semigroup with a generating element a. The ***order*** of a is defined as $\text{ord}(a) = |\langle a \rangle|$ if $\langle a \rangle$ is finite, ∞ otherwise. If there exist natural numbers k and l such that $k \neq l$ but $a^k = a^l$ then let m be the smallest natural number such that $a^m = a^{m+q}$ for some $q \in \mathbb{N}$ and let r be the smallest natural number such that $a^m = a^{m+r}$. We call m the ***index*** or ***height*** of a and r the ***period*** of a.

Theorem 3.9. *All infinite monogenic semigroups S are isomorphic to* $(\mathbb{N}, +)$. *All finite monogenic semigroups S are determined up to isomorphism by their height m and period r. In the latter case S contains a cyclic subgroup isomorphic to the residue class group $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$.*