Clifford semigroups and completely regular semigroups

Finally, we introduce two more subclasses of the class of regular semigroups.

Definition 3.40. A regular semigroup S is called a ***Clifford semigroup*** if $E(S) \subseteq C(S)$, i.e. idempotents of S commute with all elements of S.

A semigroup S is called ***completely regular*** if for every $s \in S$ there exists $x \in S$ such that $sxs = s$ and $sx = xs$. In the last case we say that $s \in S$ has a ***commuting pseudoinverse***.

Note that completely regular semigroups are exactly the unions of groups (cf. [PET77] I.5.1). Moreover, Clifford semigroups are exactly those semigroups which are inverse and completely regular (cf. [HIG92], p. 39).

On the next page we collect the results in a table for semigroups (the symbols are explained before Table I.1 on p. 30).

Semilattices of semigroups

Definition 3.41. A semigroup S is said to be a ***semilattice of its subsemigroups*** S_α, $\alpha \in Y$, if

1. Y is a semilattice;
2. $S = \dot{\bigcup}_{\alpha \in Y} S_\alpha$;
3. $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ where $\alpha\beta$ is the product of α and β in Y.

Such a decomposition of a semigroup into its subsemigroups carries a lot of information about the semigroup in the case where the subsemigroups S_α are of some specific type $\mathcal{T}$ which is better known ("simpler") than the type of S itself. If all S_α are of type $\mathcal{T}$, we say that S is a ***semilattice of semigroups of type*** $\mathcal{T}$. For more information about semilattices of semigroups see, for example, [HOW76].

Here we present (without proofs) as illustrations of the construction some structure theorems for semigroups which will be used in later chapters. The proofs of the following three theorems can be found, for example, in [HIG92] as Theorems 1.3.10, 1.3.11 and 1.3.9.

Theorem 3.42 (Clifford [Cli41]). *If a semigroup S is a union of groups, then it is a semilattice of completely simple semigroups.* □

Theorem 3.43. *A semigroup S is a Clifford semigroup if and only if it is a semilattice of groups.* □

Theorem 3.44 (Clifford [Cli41], McLean [McL54]). *A band is a semilattice of rectangular bands.* □

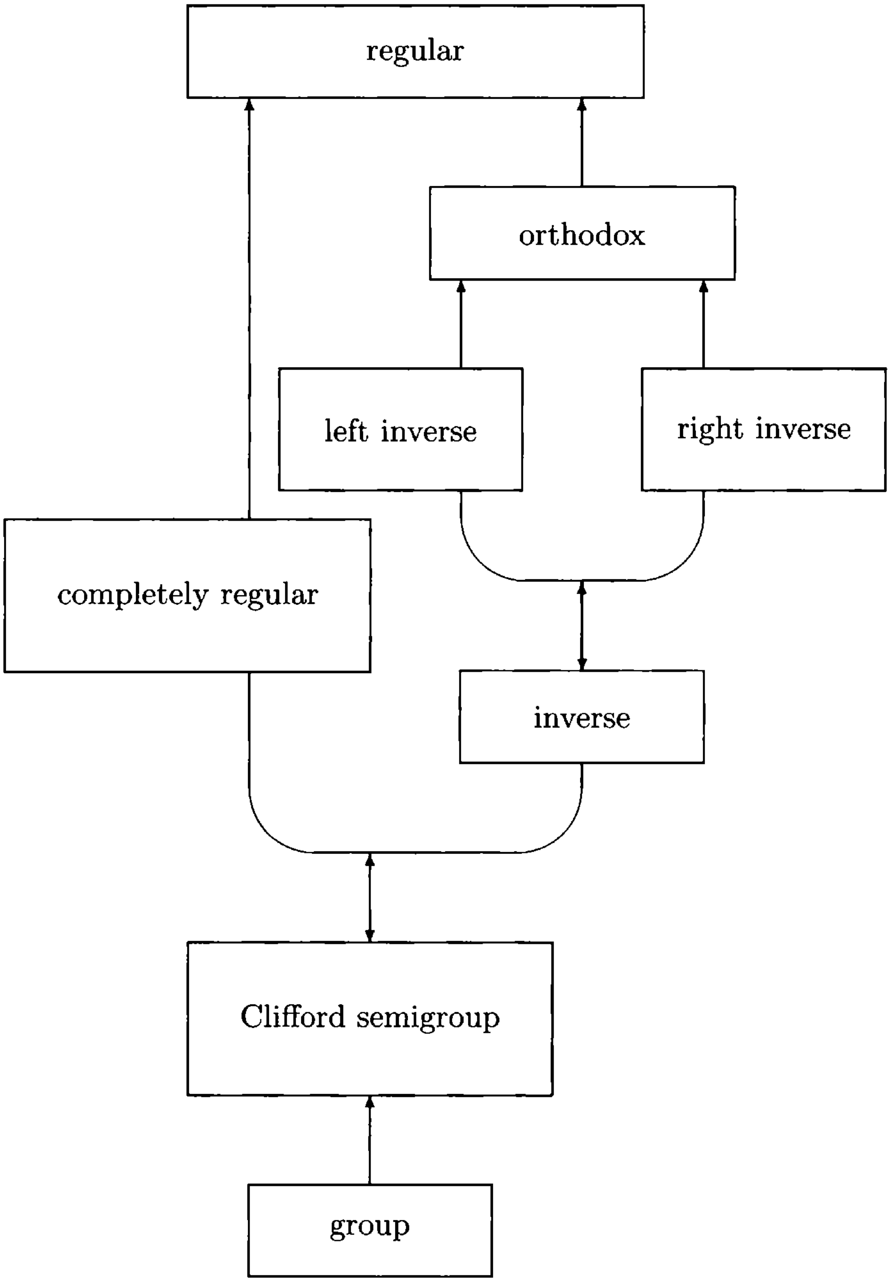

Table I.2: Regularities

We leave it to the reader to find examples which show that there are no other implications among these classes.

Right regular bands

Remark 3.45. Following Petrich [PET73] there exists the tradition to call bands which are right (left) inverse ***right (left) regular bands*** and we adopt this tradition.

Proposition 3.46. *A band is right regular if and only if it is isomorphic to a semilattice of right zero semigroups.*

Proof. Necessity. Suppose E is a right regular band. By Theorem 1.3.44 we get that E is isomorphic to a semilattice of rectangular bands E_α, $\alpha \in Y$, $E = \dot{\bigcup}_{\alpha \in Y} E_\alpha$. Let $e, f \in E_\alpha$, $\alpha \in Y$. Since we have $f(ef) = f$ in the rectangular band E_α, we have $(ef) \mathcal{L} f$. Using right regularity of E we obtain $ef = f$. Hence E_α is a right zero semigroup.

Sufficiency. Suppose E is a semilattice of right zero semigroups and $e, f \in E$. Since ef and fe belong to the same right zero semigroup we have $efe = (ef)(fe) = fe$ and thus E is right regular. □

Chain and Rees semigroups

We shall mention here some definitions and results without proofs which will be necessary only in one instant at the end of Chapter 5. Nevertheless they may give rise to additional investigations in several other places, although we will not pursue this in detail.

Definition 3.47. A semigroup is called a ***right (left) chain semigroup*** if all its right (left) congruences form a chain. A semigroup is called a ***Rees semigroup*** if every two-sided congruence on S is a Rees congruence. ***Left (right) Rees semigroups*** are defined analogously.

Construction 3.48. For $x \in \mathbb{R}$, $0 < x < 1$ we construct the following two commutative monoids.

$Q := (\{0\} \cup [x, 1], \cdot) \subseteq (\mathbb{R}, \cdot)$ with multiplication as in $\mathbb{R}$ if defined and $q_1 q_2 = 0$ if $q_1 q_2 < x$ for $q_1, q_2 \in Q$;

$R := (\{0\} \cup]x, 1], \cdot) \subseteq (\mathbb{R}, \cdot)$ with multiplication as in $\mathbb{R}$ if defined and $r_1 r_2 = 0$ if $r_1 r_2 \leq x$ for $r_1, r_2 \in \mathbb{R}$.

Definition 3.49. A monoid S with 0 is called ***atomic*** if there exists an ***atom*** $a \in S$, i.e. an element $a \in S$, $a \neq 0$, such that $ax = xa = 0$ for all $x \in S$, $x \neq 1$.

Remark 3.50. Considering the monoids from Construction 1.3.48 it is clear that Q is atomic with atom x but R is not atomic. The same is true for all submonoids of Q and R, respectively.

Theorem 3.51 (Schein ([Sch69], Theorem 1)). *Let S be a commutative semigroup and let Q, R be as in* 1.3.48. *Then S is a (left) chain semigroup if and only if one of the following conditions is satisfied:*

(1) *S is a cyclic group of order p^n or a group of type p^∞ (i.e. S is isomorphic to the multiplicative group of the roots of the polynomials $x^{p^n} - 1$ in the field $\mathbb{C}$ of complex numbers, where p is any prime number and $n \in \mathbb{N}$);*

(2) *S is as in* (1) *with a zero adjoined;*

(3) *S is a monogenic nil semigroup;*

(4) *S is as in* (3) *with an identity adjoined;*

(5) *S is an infinite subsemigroup of Q or R for some $x \in \mathbb{R}$ with the additional property that for any $y, z \in S$ with $y \leq z$ there exists $w \in S$ such that $zw = y = wz$ (such a semigroup will be called* ***order divisible****).*

In either case S can be embedded isomorphically either into Q or into R but not into both and the isomorpism is uniquely defined. □

Proposition 3.52 (Kozhukhov [Koz81]). *Let S be a non-commutative semigroup with more than two elements. Then S is a right chain semigroup if and only if one of the following conditions is satisfied:*

(1) *$S = \{e, f, u\}$ where $es = fs = s$ and $us = u$ for all $s \in S$;*

(2) *$S = \{e, u, v\}$ where $es = s$, $us = u$ and $vs = v$ for all $s \in S$;*

(3) *$S = \{e, a, u\}$ where $es = s$ and $as = us = u$ for all $s \in S$;*

(4) *$S = \{e, a, u, v, \}$ where $es = s, us = u, vs = v$ for all $s \in S$ and $a^2 = e$, $au = v$, $av = u$.* □

In [Hot69], Theorems 3.11 and 3.15, Hotzel characterizes left Rees semigroups. If one assumes that a left Rees semigroup S has an identity this characterization gives that S is commutative. Thus there is no need to distinguish between right and left Rees monoids and Hotzel's result turns into the following.

Theorem 3.53. *A monoid S is a Rees monoid if and only if it is a monogenic nil semigroup with an identity adjoined, i.e. a finite subsemigroup of some Q or R from* 1.3.48, *or fulfills condition* (5) *of Theorem* 1.3.51, *i.e. S is an infinite subsemigroup of some Q or R with the additional property that for any $y, z \in S$ with $y \leq z$ there exists $w \in S$ such that $zw = y = wz$.* □

Corollary 3.54. *Rees monoids are chain monoids.* □

Comments

In this section we went a little further in our exposition of semigroups and monoids. For more details the reader would have to consider one of several monographs on semigroups, for example, "Fundamentals of semigroups" by

J. Howie [HOW95]. We included the items which will be necessary in the sequel. Some of the easy proofs we left to the reader, for some other proofs we referred to the monographs on semigroups.

We are aware that listing many concepts and terms for later use may be tiring. To show at least some of the interconnections, we included some tables around the concepts simple and cancellative and around the concepts of regularity.

We concluded this section with some material on so-called chain and Rees semigroups where we restricted ourselves to giving definitions and citing results. This will not be essential until the very last theorem of this book. Nevertheless, these concepts may be interesting for constructing additional examples in Chapters 3 and 4 when monoids are discussed all of whose (Rees) factor acts have a certain property.

Classes of monoids will be met again under the viewpoint of homological classification, that is in particular in Chapter 4 when we describe internal properties of monoids by properties of their representations, namely their acts.

Semigroup properties of concrete endomorphism monoids have been investigated in particular for the various endomorphism monoids of graphs (cf. Definition 1.1.9). W. M. Li [Liw93], for example, directly proves that Send(G) is always regular for finite undirected graphs without loops. He determines explicitly all idempotents in each $\mathcal{L}$- or $\mathcal{R}$-class and the inverses to each element in Send(G) [Liw94c]. In [Liw95] also pseudoinverses are investigated and so are graphs for which these coincide with the inverses inside the monoid of strong endomorphisms. Similarly, completely regular elements are studied in [Liw94c]. In [Liw94a] regular elements among the endomorphisms of a finite undirected graph without loops are characterized. For Green's relations on endomorphism monoids of graphs see also [Fan95c]. In other papers bipartite graphs whose endomorphism monoid is regular [Fan93b], [Wil96], orthodox or inverse [Fan96b] are characterized. Moreover, it can be proved that bipartite graphs are even determined by their endomorphism monoids up to isomorphism [Fan96a].