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Representation of semigroups and monoids by transformations

As we have noted in Example 1.2.15(1)(c) the set $\mathcal{T}(A)$ of all transformations of a set A forms a monoid under the composition of mappings. If we write mappings on the left we get the product of $f, g : A \rightarrow A$ by the rule

$$(fg)(a) = f(g(a)) \quad \text{for all } a \in A.$$

Writing mappings on the right we get the product by the rule

$$a(fg) = (af)g \quad \text{for all } a \in A.$$

In the first case the monoid $\mathcal{T}(A)$ will be denoted by $\mathcal{T}^l(A)$ and in the second case by $\mathcal{T}^r(A)$. Apparently there exists an antiisomorphism between these monoids

$$\mathcal{T}^l(A) \rightarrow \mathcal{T}^r(A), \quad f \mapsto f, fg \mapsto gf.$$

Definition 4.1. Let S be a semigroup, A a set. A semigroup homomorphism

$$\psi : S \rightarrow \mathcal{T}^r(A)$$

is called a **representation** of S by transformations of A .

If S is a monoid and ψ is a monoid homomorphism then ψ is called a **unitary representation** of S by transformations of A .

If ψ is an antihomomorphism then ψ is called an **antirepresentation** of S by transformations of A .

Acts over semigroups and monoids

The idea of representing something by some other object which is better known at least in some respects is quite familiar in mathematics. After the excellent experiences made when representing groups by invertible matrices another important step was the representation of associative rings by endomorphisms of abelian groups. It is well known that every representation of a ring by endomorphisms of an abelian group gives a module over that ring and vice versa.

Similarly, representations of semigroups (monoids) by transformations of sets give rise to the notion of acts over semigroups (monoids).

Definition 4.2. Let S be a monoid and $A \neq \emptyset$ a set. If we have a mapping

$$\begin{aligned} \mu : A \times S &\rightarrow A \\ (a, s) &\mapsto as := \mu(a, s) \end{aligned}$$

such that

- (a) $a1 = a$ and
- (b) $a(st) = (as)t$ for $a \in A, s, t \in S$,

we call A a **right S -act** or a **right act** over S and write A_S . More informally we often say that μ defines a **right multiplication of elements** from A by elements of S .

Analogously, we define a **left S -act** A and write ${}_S A$.

Note that usually we do not specify μ .

For S -acts the following names are also common:

S-sets, *S*-operands, *S*-polygons, *S*-systems, transition systems, *S*-automata.

If S is a group or a group with 0 the name *vector system* has been used by Hoehnke [Hoe65].

The first exposition to acts over semigroups in a monograph can be found in Volume II of Clifford–Preston’s “The algebraic Theory of Semigroups” under the name of operand [CL/PR67].

Remark 4.3.

- (1) If S is a semigroup without identity, condition (a) of Definition 1.4.2 is empty. We then speak of a **semigroup act** A_S . Condition (a) may be dropped even if S is a monoid. To stress the presence of (a) one sometimes calls A_S a **unitary S -act**.
- (2) If S is a commutative monoid then every left S -act can be considered as a right S -act. Indeed, if ${}_S A$ is a left S -act we may define a right multiplication by elements of S by

$$a * s = sa$$

for $a \in A, s \in S$. Then $a * 1 = 1a = a$ for all $a \in A$ and

$$a * (s_1 s_2) = a * (s_2 s_1) = (s_2 s_1)a = s_2(s_1 a) = s_2(a * s_1) = (a * s_1) * s_2 .$$

In this situation we can consider A as a $S - S$ -biact (see Definition 1.4.24) since for all $s_1, s_2 \in S$ and $a \in A$ we have

$$(s_1 a) * s_2 = s_2(s_1 a) = (s_2 s_1)a = (s_1 s_2)a = s_1(s_2 a) = s_1(a * s_2) .$$

- (3) Every right S -act A_S can be interpreted as a left act over the **opposite monoid** S^{op} , which has the same elements as S , but multiplication \circ is defined by reversing the factors, i.e. $s \circ t = ts, s, t \in S$.
- (4) Note that the term “act” has also been used for representations of groupoids (cf. Lex and Wiegandt [Le/Wi81]).

Note that many of the properties of the acts considered firmly depend on the existence of the identity in S and condition (a) of the definition. Therefore a big portion of the structure theory developed cannot be carried over to acts over semigroups without identity.

In the sequel we shall consider mainly monoids and unitary acts over them.

Proposition 4.4. *Every unitary (anti-)representation of a monoid S by transformations in $T^r(A)$ of a set A turns A into a right (left) S -act. Conversely, for every right (left) unitary S -act A_S (${}_S A$) there is an associated (anti-)representation of S by transformations in $T^r(A)$.*

Every unitary (anti-)representation of a monoid S by transformations in $T^l(A)$ of a set A turns A into a left (right) S -act. Conversely, for every left (right) unitary S -act ${}_S A$ (A_S) there is an associated (anti-)representation of S by transformations in $T^l(A)$.

Proof. If $\psi : S \rightarrow \mathcal{T}^r(A)$ is a unitary representation of a monoid S by transformations of A , define $\mu : A \times S \rightarrow A$ by $as = \mu(a, s) := a\psi(s)$. Then μ has properties (a) and (b) of Definition 1.4.2.

Conversely, if A_S is a right S -act, i.e. we have $\mu : A \times S \rightarrow A$ with properties (a) and (b) of Definition 1.4.2, we obtain a unitary representation ψ when defining

$$\begin{aligned} \psi : S &\rightarrow \mathcal{T}^r(A) \\ s &\mapsto \mu(-, s) : a \mapsto as. \end{aligned} \quad \square$$

Semiautomata

A **semiautomaton**, i.e. an automaton without outputs becomes an act over a monoid in a natural way. As usual denote a semiautomaton by a triple (A, X, δ) where A, X are non-empty sets and $\delta : A \times X \rightarrow A$ is a mapping, A is called the *set of states*, X the *input alphabet* and δ the *next state function*. If X is a generating set of a monoid S and one has $\delta(a, xx') = \delta(\delta(a, x), x')$ for any $a \in A, x, x' \in X$ then S is called the **input monoid** of (A, X, δ) .

Proposition 4.5. *Semiautomata can be considered as S -acts and S -acts can be considered as semiautomata (with a possibly non-free input monoid).*

Proof. If (A, X, δ) is a semiautomaton, then obviously A can be viewed as a right X^* -act where X^* denotes the free monoid generated by the elements of X (cf. Example 1.2.15 (1)(f)), when extending $\delta : A \times X \rightarrow A$ to $\delta^* : A \times X^* \rightarrow A$ by setting $\delta^*(a, xx') = \delta(\delta(a, x), x')$.

Conversely, an S -act is a semiautomaton with input monoid S which may or may not be free, μ of Definition 1.4.2 being the next state function. \square

Consequently, we might as well speak of semiautomata instead of S -acts throughout this book.

Presentation of finite S -acts

Construction 4.6. (a) Following the pattern common for automata and, as well, the idea of Cayley color graphs for groups and semigroups (cf. Remark 1.2.35) we can represent a finite S -act A_S where $A = \{a_1, \dots, a_m\}$ and $S \setminus \{1\}$ with a set of generating elements $\{x_1, \dots, x_n\}$ by a directed graph with colored edges as follows. Take a graph with the vertex set A and colors x_1, \dots, x_n , such that (a_i, a_j) is an edge of color x_k if $a_i x_k = a_j$.

(b) In analogy with groupoids we can present a left S -act ${}_S A$ by a *multiplication table* where the elements of S are in the left margin and the elements of A in the top line if S and A are finite. Examples can be found on p. 48.

Injective and surjective action

Definition 4.7. We say that an element $s \in S$ *acts injectively* on A_S if $as = a's$, $a, a' \in A_S$, implies $a = a'$. If every $s \in S$ acts injectively on A_S then we say that S *acts injectively* on A_S .

We say that an element $s \in S$ *acts surjectively* on A_S if $A_Ss = \{as \mid a \in A_S\} = A_S$. If every $s \in S$ acts surjectively on A_S then we say that S *acts surjectively* on A_S .

Faithful and strongly faithful acts

We now define faithful action, which in the language of automata means that different inputs operate differently on at least one state. An action is strongly faithful in this interpretation if different inputs operate differently everywhere.

Definition 4.8. We call A_S a *faithful* right S -act if for $s, t \in S$ the equality $as = at$ for all $a \in A$ implies $s = t$.

We call A_S a *strongly faithful* S -act if for $s, t \in S$ the equality $as = at$ for some $a \in A$ implies that $s = t$.

It is obvious that every strongly faithful act is faithful. The acts S_S and ${}_S S$ (see Example 1.4.9(1)) are always faithful since $1 \in S$ and they are strongly faithful if S is left or right cancellative, respectively.

It is clear from the definitions that if an act contains a faithful subact then it is itself also faithful and if A_i , $i \in I$, are strongly faithful right S -acts, then $\bigcup_{i \in I} A_i$ is also a strongly faithful act.

Examples of S -acts

Example 4.9.

- (1) Take any monoid S . Let K be a right ideal of S . Since $ks \in K$ for any $k \in K$ and $s \in S$ we have that K is a right S -act. To stress this we write K_S . In particular, S_S (resp. ${}_S S$) is a right (resp. left) S -act and even a biact ${}_S S_S$ (see definition 1.4.24). If R is a submonoid of S then ${}_R S_R$ is an $R - R$ -biact in the obvious way.
- (2) If S is the one-element monoid $\{1\}$, then trivially every non-empty set X is a left and right S -act.
- (3) Let $(K, +, \cdot)$ be a field and V a left K -vector space. Then V is a left (K, \cdot) -act, but not a $(K, +)$ -act.