**3. Sequences**

(3.1) **Definition**: Let be a non-empty set. A function which its domain and its codomain is called a sequence in, such that if .

(3.2) **Example:** If be a sequence defined in a range is .

(3.3) **Definition**: Let be a sequences in , we say that is a subsequence of , if there is a function

1. ;
2. .

(3.4) **Example:** Let , we note that is a subsequence of , since if we define by, and then be a subsequence of {.

(3.5) **Note:** If is a subsequence of and is a subsequence of, then is a subsequence of .

(3.6) **Definition**: If be a sequence in a partially ordered set , we say that be an increasing, if , and we say that be a decreasing, if and we say that be a monotone, if an increasing or a decreasing.

(3.7) **Note:**

* be an increasing.
* be an increasing and sup , .
* be a decreasing.
* be a decreasing and inf , .

(3.8) **Definition**: Let be a sequence in a partially ordered set , we say that is a converges to , if there is in , such that

1. ;
2. and .

(3.9) **Note:** is called a converge point and written .

(3.10) **Definition**: Let be a sequence in a partially ordered set , we have

* Inferior limit lim inf , where lim inf .
* Superior limit lim sup , where lim sup .

(3.11) **Note:** If lim sup lim sup .

**Real Sequences**

(3.12) **Note:** We saythat be a real sequence if .

(3.13) **Definition**: Arithmetic progression is a sequence which be subtract output of every term from direct previous term is equal to constant called progression basis and denoted by .

(3.14) **Example:** Arithmetic progression which its first term and its basis is

. The general term of arithmetic progression is where represents a first term and represents a basis with the partial summation

.

(3.15) **Definition**: Geometry progression is a sequence which output of division of every term on direct previous term is equal to a constant called progression basis and denoted by .

(3.16) **Example:** Geometric progression which its first term and its basis is

. The general term is where represents a first term and represents a basis with the partial summation

.

If .

If .

(3.17)**Definition**: Arithmetic geometric progression is . The general term is and the partial summation is

.

If .

(3.18) **Definition**: Let be a real sequence, we say that

1. Convergent, if , we say that a point is a limit point of and its written by or where , therefore iff .
2. Divergent, if does not convergent.
3. Cauchy sequence, if and then is a Cauchy sequence iff where .

(3.19) **Examples:**

1. Show that .

**Solution:** since .

1. Show that .

**Solution:** since (by Archimedes property), , so .

1. Show that be a divergent.

**Solution:** since if we assume that be a convergent and then contains of terms , since (by Archimedes property) , since , this means does not contain on terms of , but this is contradiction.

1. Show that such that be a convergent and converges to one .

**Solution:** since , take and then .

1. Show that be a divergent.

**Solution:** since if we suppose that be a convergent , let .

Let , take is an even, is an odd, this means that does not contain all terms of and then does not converge to .

By same way we prove that does not converge to .

Now, let , let , take , we deduce that does not contain on any term of does not converge to .

(3.20) **Theorem:**

1. If a real sequence is a convergent, then a converge point is a unique.
2. Every convergent sequence be Cauchy sequence.

**Proof:** (1) Let and let , since put max , but this is a contradiction .

(2) let be a convergent sequence , let , since , if and then be Cauchy sequence.

(3.21) **Definition**: If be a real sequence, we say that is

1. Bounded above, if ;
2. Bounded below, if ;
3. Bounded, if .

(3.22) **Examples**:

1. is a bounded, since .
2. is a bounded, since .
3. is a bounded, since .
4. does not bounded, since if we suppose that is a bounded , but this is a contradiction (Archimedes property) since .
5. does not bounded.

(3.23) **Theorem**: Every Cauchy sequence be a bounded, and then every convergent sequences be a bounded.

**Proof:** Let be Cauchy sequence, we must prove that is a bounded. Let , since is a Cauchy sequence , let , since, put max, and then is a bounded.

(3.24) **Note**: If a real sequence is a bounded, then its not a necessary be a convergent, for example is a bounded, but does not convergent.

(3.25) **Definition**: Let be a real sequence. We said that

1. Non-decreasing, if .
2. Increasing, if .
3. Non-increasing, if .
4. Decreasing, if .

(3.26) **Note**: We said that is a monotonic, if its be satisfy any one of above.

(3.26) **Examples**:

1. { is a decreasing a monotonic.
2. is an increasing a monotonic.
3. does not a monotonic.

(3.27) **Theorem**:

1. Every bounded real sequence and monotonic be a convergent.
2. Every bounded real sequence contains on a convergent partial sequence.

(3.28) **Theorem**: (Some special sequences)

1. If .
2. If .
3. .
4. If .

**Proof:** (1) let , take .

(2) a. if , put , since .

b. if .

c. if , put and , since .

(3) let , since .

(4) .

a. if .

b. if exists, put , since , , put .

(3.29) **Theorem**: Let be a real sequences such that and , then

1. .
2. .
3. .
4. .
5. where .
6. where .
7. .
8. .
9. If .

**Proof:** (1) let , since , since , put max , , so .

(3.30) **Theorem**:

1. For all real number, there is Cauchy sequence of rational numbers converge of them.
2. For all real number, there is Cauchy sequence of irrational numbers converge of them.
3. There is Cauchy sequence of rational numbers does not converge to any rational number.

**Proof:** (1) let , since (by density of rational numbers) , now, we must prove that , let (by Archimedes property) .

(3.31) **Definition**: We said that a space is a complete, if every Cauchy sequence in be a convergent in .

(3.32) **Note**: is an incomplete, while is a complete.