**4. Infinite Series**

Let $\{a\_{n}\}$ a real sequence and $S\_{1}=a\_{1}, S\_{2}=a\_{1}+a\_{2},S\_{3}=a\_{1}+a\_{2}+a\_{3}, …, S\_{n}=a\_{1}+a\_{2}+…+a\_{n}$. A sequence of partial sums $\{S\_{n}\}$ is called an infinite series and its denoted by $\sum\_{n=1}^{\infty }a\_{n}$. We say that $a\_{1}, a\_{2}, a\_{3}, …$ be a terms of infinite series $\sum\_{n=1}^{\infty }a\_{n}$ and called of numbers $S\_{1}, S\_{2}, S\_{3},…$ be a partial sums of infinite series $\sum\_{n=1}^{\infty }a\_{n}$.

(4.1) **Definition**: Let $\{a\_{n}\}$ be a real sequence and $S\_{n}=\sum\_{k=1}^{n}a\_{k}$, we called of $\{S\_{n}\}$ is an infinite series and denoted by $\sum\_{n=1}^{\infty }a\_{n}$.

(4.2) **Definition**: We say that $\sum\_{n=1}^{\infty }a\_{n}$ is a convergent, if $\{S\_{n}\}$ is a converge to $S$, this means ($\lim\_{n\to \infty }S\_{n}=S$), $S$ is called infinite series sum $\sum\_{n=1}^{\infty }a\_{n}$, this means $S=\sum\_{n=1}^{\infty }a\_{n}$. If $\{S\_{n}\}$ is a divergent (i.e. $\lim\_{n\to \infty }S\_{n}$ does not exist).

(4.3) **Example**: Does $\sum\_{n=1}^{\infty }\frac{1}{n(n+1)}$ convergent$ ?$

$a\_{n}=\frac{1}{n(n+1)}, S\_{n}=\sum\_{k=1}^{n}a\_{k}=\sum\_{k=1}^{n}\frac{1}{k(k+1)}=\sum\_{k=1}^{n}(\frac{1}{k}-\frac{1}{k+1})=1-\frac{1}{n+1}⟹S\_{n}⟶1⟹\sum\_{n=1}^{\infty }\frac{1}{n\left(n+1\right)}=1$ and then $\sum\_{n=1}^{\infty }\frac{1}{n\left(n+1\right)}$ is a convergent.

(4.4) **Theorem**: (some special infinite series)

1. $\sum\_{n=1}^{\infty }ar^{n-1}\ni a\ne 0, r\ne 0$ is called geometric series and $r$ is a basis of series. $\sum\_{n=1}^{\infty }ar^{n-1}$ is a convergent, if $\left|r\right|<1, S=\frac{a}{1-r}$ and otherwise its be a divergent.
2. $\sum\_{n=1}^{\infty }\frac{1}{n}$ is called a harmonic series and it’s a divergent.

**Proof:** (1) if $r=1⟹S\_{n}=a+a+…+a=na⟹\{na\}$ does not convergent, if it’s a convergent, so it’s a bounded, this means $∃ M\in R^{+}\ni \left|na\right|\leq M ∀n\in Z^{+}⟹n\left|a\right|\leq M⟹n\leq \frac{M}{\left|a\right|} ∀n\in Z^{+}$, but this a contradiction (Archimedes property) $⟹\sum\_{n=1}^{\infty }ar^{n-1}$ is a divergent.

(2) $a\_{n}=\frac{1}{n}, S\_{n}=\sum\_{k=1}^{n}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+…+\frac{1}{n}, S\_{2n}=\sum\_{k=1}^{2n}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+…+\frac{1}{n}+\frac{1}{n+1}+…+\frac{1}{2n}⟹S\_{2n}-S\_{n}\geq \frac{1}{2} ∀n\in Z^{+}$, i.e. if $m=2n, n\geq 1⟹\left|S\_{m}-S\_{n}\right|\geq \frac{1}{2}∀n,m\in Z^{+}⟹\{S\_{n}\}$ does not Cauchy sequence $⟹\{S\_{n}\}$ does not convergent $⟹\sum\_{n=1}^{\infty }\frac{1}{n}$ is a divergent.

(4.5) **Examples**:

1. $\sum\_{n=0}^{\infty }\frac{1}{2^{n}}$ is a convergent, since $r=\frac{1}{2}$ and $\sum\_{n=1}^{\infty }\frac{1}{2^{n}}=2$.
2. $\sum\_{n=1}^{\infty }4^{n-1}$ is a divergent, since $r=4$.
3. $\sum\_{n=1}^{\infty }(-\frac{1}{6})^{n-1}$ is a convergent, since $r=-\frac{1}{6}$ and $\sum\_{n=1}^{\infty }(-\frac{1}{6})^{n-1}=\frac{6}{7}$ .
4. $0.1+0.01+0.001+…$ is a convergent, since $0.1+0.01+0.001+…=\frac{1}{10}+\frac{1}{10^{2}}+\frac{1}{10^{3}}+…=\sum\_{n=1}^{\infty }\frac{1}{10^{n}}=\sum\_{n=1}^{\infty }\frac{1}{10}.\frac{1}{10^{n-1}}⟹a=\frac{1}{10}, r=\frac{1}{10}⟹\sum\_{n=1}^{\infty }\frac{1}{10^{n}}=\frac{1}{9}$ .
5. The number $0.16666…$ is a convergent, let $0.16=0.16666…=0.1+0.06+0.006+0.0006+…=0.1+\sum\_{n=1}^{\infty }\frac{1}{10^{n+1}}=\sum\_{n=1}^{\infty }\frac{6}{100}.\frac{1}{10^{n+1}}⟹a=\frac{6}{100}, r=\frac{1}{10}⟹0.16=0.1+\sum\_{n=1}^{\infty }\frac{1}{10^{n+1}}=\frac{1}{6}$ .

(4.6) **Theorem**: Let $\sum\_{n=1}^{\infty }a\_{n}$ and $\sum\_{n=1}^{\infty }b\_{n}$ be a convergent infinite series, then

1. $\sum\_{n=1}^{\infty }(a\_{n}+b\_{n})$ is a convergent and $\sum\_{n=1}^{\infty }(a\_{n}+b\_{n})=\sum\_{n=1}^{\infty }a\_{n}+\sum\_{n=1}^{\infty }b\_{n}$.
2. $\sum\_{n=1}^{\infty }λa\_{n}$ is a convergent $∀λ\in R$ and $\sum\_{n=1}^{\infty }λa\_{n}=λ\sum\_{n=1}^{\infty }a\_{n}$.

**Proof:** (1) Let $S\_{n}=\sum\_{k=1}^{\infty }a\_{k}$ and $T\_{n}=\sum\_{k=1}^{\infty }b\_{k}$, since $\sum\_{n=1}^{\infty }a\_{n}$ and $\sum\_{n=1}^{\infty }b\_{n}$ be a convergent infinite series $⟹$ $\sum\_{n=1}^{\infty }a\_{n}=S, \sum\_{n=1}^{\infty }b\_{n}=T⟹\{S\_{n}\}, \{T\_{n}\} $be a convergent $⟹S\_{n}\rightarrow S, T\_{n}\rightarrow T ⟹S\_{n}+T\_{n}\rightarrow S+T$, $S\_{n}+T\_{n}=\sum\_{n=1}^{\infty }(a\_{n}+b\_{n})\rightarrow S+T⟹\sum\_{n=1}^{\infty }(a\_{n}+b\_{n})=S+T=\sum\_{n=1}^{\infty }a\_{n}+\sum\_{n=1}^{\infty }b\_{n}$ .

(4.7) **Corollary**: If $\sum\_{n=1}^{\infty }a\_{n}$ is a convergent and $\sum\_{n=1}^{\infty }b\_{n}$ is a divergent, then

1. $\sum\_{n=1}^{\infty }(a\_{n}+b\_{n})$ is a divergent.
2. $\sum\_{n=1}^{\infty }λb\_{n}$ is a divergent $∀λ\ne 0$.

**Proof:** (1) Suppose that $\sum\_{n=1}^{\infty }(a\_{n}+b\_{n})$ is a convergent, since $\sum\_{n=1}^{\infty }a\_{n}$ is a convergent $⟹ -\sum\_{n=1}^{\infty }a\_{n}$ is a convergent.

Since $\sum\_{n=1}^{\infty }b\_{n}=\sum\_{n=1}^{\infty }(a\_{n}+b\_{n}-a\_{n})$ is a convergent, but this is a contradiction.

(4.8) **Example**: $\sum\_{n=1}^{\infty }\frac{1}{n}$ and $-\sum\_{n=1}^{\infty }\frac{1}{n}$ are a divergent, but $\sum\_{n=1}^{\infty }(\frac{1}{n}-\frac{1}{n})=\sum\_{n=1}^{\infty }0$ is a convergent.