**12. Continuity**

 (12.1)**Definition**: If $\left(X,d\_{1}\right),(Y,d\_{2})$ be metric spaces. We said that a function $f:X\rightarrow Y$ is continuous at $x\_{0}\in X$, if $∀$ open set $U⊆Y$ contains $f\left(x\_{0}\right),∃$ an open set $V⊆X$ contains $x\_{0}\ni f(V)⊂U$.

(12.2)**Theorem:** Let $\left(X,d\_{1}\right),(Y,d\_{2})$ be metric spaces, then a function $f:X\rightarrow Y$ be continuous at point $x\_{0}\in X⟺∀$ open ball $B\_{ε}(f\left(x\_{0}\right))$ in $Y ∃$ an open ball $B\_{δ}(x\_{0})$ in $X\ni f(B\_{δ}\left(x\_{0}\right))⊂B\_{ε}(f\left(x\_{0}\right))$.

This means, $∀ε>0 ∃ δ>0\ni ∀x\in X⟹d\_{1}(x,x\_{0})<δ⟹d\_{2}(f\left(x\right),f(x\_{0}))<ε$.

(12.3)**Example:** Let $(R,d\_{u})$ be usual metric space. Prove that a function $f:R\rightarrow R$ defined by $f\left(x\right)=x^{2}, x\in R$ is a continuous.

**Solution:** let $x\_{0}\in R, ε>0$.

$$\left|f\left(x\right)-f( x\_{0})\right|=\left|x^{2}-x\_{0}^{2}\right|=\left|(x-x\_{0})(x+x\_{0})\right|=\left|x-x\_{0}\right|\left|x+x\_{0}\right|$$

Since $\left|x+x\_{0}\right|\leq \left|x\right|+\left|x\_{0}\right|$

So, $\left|f\left(x\right)-f( x\_{0})\right|\leq \left|x-x\_{0}\right|(\left|x\right|+\left|x\_{0}\right|)…(1)$

Since $x=\left(x-x\_{0}\right)+x\_{0}⟹\left|x\_{0}\right|=\left|(x-x\_{0})+x\_{0}\right|⟹\left|x\right|\leq \left|x-x\_{0}\right|+\left|x\_{0}\right|$

$$⟹\left|x\right|-\left|x\_{0}\right|\leq \left|x-x\_{0}\right|$$

If $\left|x-x\_{0}\right|<1⟹\left|x\right|-\left|x\_{0}\right|<1⟹\left|x\right|<1+\left|x\_{0}\right|…(2)$

From $\left(1\right), (2)$, we get

$$\left|f\left(x\right)-f( x\_{0})\right|\leq \left|x-x\_{0}\right|(1+2\left|x\_{0}\right|)…(3)$$

Take $δ=$ min $\{1,\frac{ε}{1+2\left|x\_{0}\right|}\}$

Now, let $x\in R\ni \left|x-x\_{0}\right|<δ⟹\left|x-x\_{0}\right|<\frac{ε}{1+2\left|x\_{0}\right|} $ and $\left|x-x\_{0}\right|<1$

$$⟹\left|x-x\_{0}\right|<(1+2\left|x\_{0}\right|)<ε$$

From $(3)$, we get

$$\left|f\left(x\right)-f( x\_{0})\right|<ε$$

$⟹f$ is a continuous at $x\_{0}⟹f$ is a continuous.

(12.4)**Example:** Let $(R,d\_{u})$ be usual metric space. Prove that a function $f:R\rightarrow R$ defined by $f\left(x\right)=\frac{1}{x}, x\in R^{+}$ is a continuous at $x=2$.

**Solution:** let $ε>0,\left|f\left(x\right)-f( 2)\right|=\left|\frac{1}{x}-\frac{1}{2}\right|=\left|\frac{2-x}{2x}\right|=\frac{\left|2-x\right|}{2x}$ (since $x>0$)

If $\left|x-2\right|<1⟹-1<x-2<1⟹1<x<3⟹x>1⟹\frac{1}{x}<1$

$$⟹\frac{\left|2-x\right|}{2x}<\frac{1}{2}\left|2-x\right|$$

Choose $δ=$ min $\{1,2ε\}$

Now, let $x\in R^{+}\ni \left|x-2\right|<δ⟹\left|x-2\right|<2ε$ and $\left|x-2\right|<1$

$⟹\frac{1}{2}\left|x-2\right|<ε$

$\left|f\left(x\right)-f( 2)\right|<\frac{1}{2}\left|x-2\right|<ε$

$⟹f$ is a continuous at $x=2$.

(12.5)**Example:** Let $(R,d\_{u})$ be usual metric space. Prove that a function $f:R\rightarrow R$ defined by $f\left(x\right)=\left\{\begin{array}{c}1, x>0\\0, x=0\\-1, x<0\end{array}\right., $ is a continuous on $R\\{0\}$.

(12.6)**Theorem:** Let $\left(X,d\_{1}\right),(Y,d\_{2})$ be metric spaces, and $f:X\rightarrow Y$ a function, then the following properties are equivalent:

1. A function $f$ is a continuous.
2. If an open set $G⊂Y$, then $f^{-1}(G)$ be an open set in $X$.
3. If a closed set $H⊂Y$, then $f^{-1}(H)$ be a closed set in $X$.
4. $f\left(\overbar{A}\right)⊆\overbar{f\left(A\right)} ∀A⊂X$.
5. $\overbar{f^{-1}\left(B\right)}⊆f^{-1}(\overbar{B}) ∀B⊂Y$.
6. $f^{-1}\left(B^{°}\right)⊆(f^{-1}\left(B\right))^{°}$.

(12.7)**Theorem:** Let $\left(X,d\_{1}\right),\left(Y,d\_{2}\right),\left(Z,d\_{3}\right)$ be metric spaces, and $f:X\rightarrow Y,g:Y\rightarrow Z$ be a continuous functions, then a function $g∘f:X\rightarrow Z$ be a continuous function.

**Proof:** let $G$ is an open set in $Z$.

Since a function $g:Y\rightarrow Z$ is a continuous $⟹g^{-1}(G)$ is an open set in $Y$.

Since a function $f:X\rightarrow Y$ is a continuous $⟹f^{-1}(g^{-1}(G))$ is an open set in $X$, but $f^{-1}(g^{-1}(G))=(f^{-1}∘g^{-1})(G)=(g∘f)^{-1}(G)$

$⟹(g∘f)^{-1}(G)$ is an open set in $X$

$⟹g∘f$ be a continuous function.

(12.8)**Example:** Let $\left(X,d\_{1}\right),(Y,d\_{2})$ be metric spaces, and $f:X\rightarrow Y$ a function. Prove that

1. If $f$ is a constant, then $f$ is a continuous.
2. If $\left(X,d\_{1}\right)$ be discrete, then $f$ is a continuous.
3. If $\left(X,d\_{1}\right)$ be indiscrete, then $f$ is a continuous.

**Solution:** (1) since $f$ is a constant, then $∃ b\in Y\ni f\left(x\right)=b ∀x\in X$.

Let $G$ be an open set in $Y$.

$$f^{-1}\left(G\right)=\left\{\begin{array}{c}∅, b\notin G\\X, b\in G\end{array}\right.$$

Since $∅, X$ be an open sets $⟹f^{-1}\left(G\right)$ be an open set in $X⟹f$ is a continuous.

**Sequentially Continuity**

(12.9)**Definition:** Let $\left(X,d\_{1}\right),(Y,d\_{2})$ be metric spaces. We said a function $f:X\rightarrow Y$ be sequentially continuity at $x\in X$, if every sequence $\{x\_{n}\}$ in $X\ni x\_{n}\rightarrow x\_{0}⟹f(x\_{n})\rightarrow f(x\_{0})$ in $Y$.

(12.10)**Theorem:** Let $\left(X,d\_{1}\right),(Y,d\_{2})$ be metric spaces, then a function $f:X\rightarrow Y$ be a continuous at $x\_{0}\in X⟺f$ be sequentially continuity at $x\_{0}\in X$.

(12.11)**Example:** Let $(R,d\_{u})$ be usual metric space. Prove that a function $f:R\rightarrow R$ defined by $f\left(x\right)=\left\{\begin{array}{c}1, x>0\\0, x=0\\-1, x<0\end{array}\right., $ is a discontinuous at $x=0$.

**Solution:** take $x\_{n}=\frac{1}{n}⟹\{x\_{n}\}$ in $R$ and $x\_{n}\rightarrow 0$,

since $\frac{1}{n}>0 ∀n\in Z^{+}⟹f\left(\frac{1}{n}\right)=1⟹f\left(x\_{n}\right)=1$

$$⟹f\left(x\_{n}\right)\rightarrow 1⟹f\left(x\_{n}\right)↛f\left(0\right)=0$$

$⟹f$ is a discontinuous at$ x=0.$