1. **Measure**

**Lengths of Bounded Open Sets**

(10.1) **Definition**: Let $I$ bounded open interval in $R$, this means $I=∅$ or

$$I=\left(a,b\right)=\{x\in R:a<x<b\}$$

Length of $I$ denoted by $L(I)$ and defined as $L\left(I\right)=\left\{\begin{array}{c}b-a,I=(a,b) \\0, I=∅\end{array}\right.$

(10.2) **Examples:**

1. If $I=(3,8)$, then $L\left(I\right)=8-3=5$.
2. If $I=(-2,6)$, then $L\left(I\right)=6-(-2)=8$.
3. If $I=(-4,-2)$, then $L\left(I\right)=-2-(-4)=2$.

(10.3) **Theorem:** If $I\_{1 }, I\_{2 }$bounded open intervals and $I\_{1 }⊆ I\_{2 }$, then $L(I\_{1 })\leq L(I\_{2 })$.

**Proof:** let $I\_{1 }=\left(a\_{1}, b\_{1}\right), I\_{2 }=\left(a\_{2}, b\_{2}\right)$

Since $I\_{1 }⊆ I\_{2 }$

$$⟹a\_{2}\leq a\_{1} , b\_{1}\leq b\_{2} $$

$$⟹b\_{1}-a\_{1}\leq b\_{2}-a\_{1}\leq b\_{2}-a\_{2}$$

$⟹L(I\_{1 })\leq L(I\_{2 })$.

(10.4) **Theorem:** If $I,I\_{1 }, I\_{2 },…, I\_{n }$bounded open intervals and $I⊆ \bigcup\_{k=1}^{n}I\_{k }$, then $L(I)\leq \sum\_{k=1}^{n}L(I\_{k })$.

**Proof:** by mathematical induction .

When $n=1⟹I⊆I\_{1 }⟹L(I)\leq L(I\_{2 })$.

Assume that when $n=r$ , then the relation is true , this means

$$I⊆ \bigcup\_{k=1}^{r}I\_{k }⟹L(I)\leq \sum\_{k=1}^{r}L(I\_{k })$$

Now, we must prove that the relation is true where $n=r+1$

$$I⊆ \bigcup\_{k=1}^{r+1}I\_{k }=\bigcup\_{k=1}^{r}I\_{k }∪I\_{r+1 }$$

$$L(I)\leq L(\bigcup\_{k=1}^{r}I\_{k }∪I\_{r+1 })\leq L((\bigcup\_{k=1}^{r}I\_{k })+I\_{r+1 })$$

$$\leq \sum\_{k=1}^{r}L\left(I\_{k }\right)+L\left(I\_{r+1 }\right)=\sum\_{k=1}^{r+1}L(I\_{k })$$

Therefore, the relation is true where $n=r+1$

$⟹$ the relation true for all $n\in Z^{+}$.

(10.5) **Theorem:** If $I,I\_{1 }, I\_{2 },…,$bounded open intervals and $I⊆ \bigcup\_{k=1}^{\infty }I\_{k }$, then $L(I)\leq \sum\_{k=1}^{\infty }L(I\_{k })$.

**Proof:** since $I$ bounded open interval

$$⟹I=∅ or I=\left(a,b\right)$$

If $I=∅⟹$ the proof is clear.

If $I=\left(a,b\right)$

Let $ε>0$, and let $I\_{0 }=\left(b-ε,b+ε\right), I\_{-1 }=\left(a-ε,a+ε\right)$

$⟹$ a family $\{I\_{k }:k\geq -1\}$ open cover of $[a,b]$

Since $[a,b]$ compact set $⟹I⊆I\_{-1 }∪I\_{0 }∪…∪I\_{n }$

$$L\left(I\right)\leq L\left(I\_{-1 }\right)+L\left(I\_{0 }\right)+L\left(I\_{1 }\right)+…+L\left(I\_{n }\right)+…$$

$$=L\left(I\_{-1 }\right)+L\left(I\_{0 }\right)+\sum\_{k=1}^{\infty }L(I\_{k })$$

$$=2ε+2ε+\sum\_{k=1}^{\infty }L\left(I\_{k }\right)\leq 4ε+\sum\_{k=1}^{\infty }L(I\_{k })$$

Since $ε$ arbitrary $⟹L(I)\leq \sum\_{k=1}^{\infty }L(I\_{k })$.

(10.6) **Theorem:** If $\left\{I\_{n }\right\},\{J\_{k }\}$ countable family of bounded open intervals $\ni \bigcup\_{n=1}^{\infty }I\_{n }⊆\bigcup\_{k=1}^{\infty }J\_{k }⟹\sum\_{n=1}^{\infty }L(I\_{n })\leq \sum\_{k=1}^{\infty }L(J\_{k })$.

**Proof:** $I\_{n }∩J\_{k }$bounded open interval $∀n,k$

$$∀n, I\_{n }=\bigcup\_{k=1}^{\infty }(I\_{n }∩J\_{k })$$

$$⟹L(I\_{n })\leq \sum\_{k=1}^{\infty }L(I\_{n }∩J\_{k })$$

$$\bigcup\_{k=1}^{\infty }(I\_{n }∩J\_{k })=(\bigcup\_{n=1}^{\infty }I\_{n })∩J\_{k }⊆J\_{k }$$

$$\sum\_{n=1}^{\infty }L(I\_{n }∩J\_{k })\leq L(J\_{k })$$

$$\sum\_{n=1}^{\infty }L(I\_{n })\leq \sum\_{n=1}^{\infty }\sum\_{k=1}^{\infty }L\left(I\_{n }∩J\_{k }\right)=\sum\_{k=1}^{\infty }\sum\_{n=1}^{\infty }L\left(I\_{n }∩J\_{k }\right)\leq \sum\_{k=1}^{\infty }L(J\_{k })$$

(10.7) **Definition**: Let $G$ open bounded set in $R$, we have

$$G=\bigcup\_{n=1}^{\infty }I\_{n }, I\_{n }∩I\_{m }=∅ ∀n\ne m$$

Where $\{I\_{n }\}$ bounded open intervals.

Length of set $G$ denoted by $L(G)$ and defined as

$$L\left(G\right)=\sum\_{n=1}^{\infty }L(I\_{n })$$

(10.8) **Theorem**: Let $F$ family of all bounded open sets in $R$, then $L:F\rightarrow R$ function and we have

1. $L\left(A\right)\geq 0 ∀A\in F$.
2. $L\left(∅\right)=0$.
3. If $A,B\in F\ni A⊆B⟹L\left(A\right)\leq L\left(B\right)$.
4. If $A,B\in F⟹L\left(A∪B\right)+L\left(A∩B\right)=L\left(A\right)+L\left(B\right)⟹L\left(A∪B\right)\leq L\left(A\right)+L\left(B\right)$ and if $A∩B=∅⟹L\left(A∪B\right)=L\left(A\right)+L\left(B\right)$.
5. Let $\{A\_{n}\}$ sequence of sets in $F\ni \bigcup\_{n=1}^{\infty }A\_{n}\in F⟹L(\bigcup\_{n=1}^{\infty }A\_{n})\leq \sum\_{n=1}^{\infty }L(A\_{n})$.
6. If $A\in F, r\in R⟹L\left(A\right)= L\left(A+r\right)\ni A+r=\{x+r:x\in A\}$.