الجامعة المستنصريـــــــــــة / كليـــــــــة العلوم

قســــــــم الفيزيـــــــــاء

Mustansiriyah University

College of Science

Physics Department

Lecture (1) for PhD

Edited by:

- **Prof. Ali .A. Al-Zuky**
- **Bahaa Jawad**
- **Saif Mohamed**
- **Sanaa Thamir**

Wave equation

The general formula of differential wave equation given by:

$$\frac{\partial^2\psi}{\partial t^2} = V^2 \frac{\partial^2\psi}{\partial x^2} \text{..........................}(1)$$

Or

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2} \text{..........................}(2)$$

When $\psi\ (x,t)$ wave equation for wave travel in speed (V). Use partial differential equation to solve eq.(1) or (2):

Suppose : $\psi = Q(x)R(t)$ ....(3)

So eq.(1) becomes:

$$\frac{1}{R}\frac{\partial^2 R}{\partial t^2} = \frac{V^2}{Q}\frac{\partial^2 Q}{\partial x^2} = A = separation\ constan \text{ ....}(4)$$

So get:

$$\frac{1}{R}\frac{\partial^2 R}{\partial t^2} = A$$

$$\frac{\partial^2 R}{\partial t^2} = AR(t) \text{......................}(5)$$

$$\frac{\partial^2 Q}{\partial x^2} = \frac{A}{V^2} Q(x) \text{ ....}(6)$$

Now need to solve eq.(5) & (6):

Case (1): when A=0

$\therefore$ $$\frac{\partial^2 R}{\partial t^2} = 0$$

$$\frac{\partial R}{\partial t} = a = constant$$

$$R = at + b \text{..............(7)}$$

&

$$\frac{\partial^2 Q}{\partial x^2} = 0$$

$$\frac{\partial Q}{\partial x} = c = constant$$

$Q = Cx + d$...............(8)

When a,b,c &d are constant.

$\therefore \quad \psi = Q.R$ = (at+b)(cx+d)

$$\psi = a_1 + b_1 x + c_1 t + d_1 xt \text{ ....(9)}$$

Case (2): when A = $\omega^2 > 0$ get:

$$\frac{\partial^2 R}{\partial t^2} = AR(t) = \omega^2 R(t) \text{ ....(10)}$$

&

$$\frac{\partial^2 Q}{\partial x^2} = (\frac{\omega}{V})^2 Q \text{ ....(11)}$$

- H.W (1): find $\psi(x, t)$

Case (3): when A = $-\omega^2 < 0$: H.W(2) find $\psi(x, t)$?!

- H.W (3) : Solve the following wave eq.

$$9 \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial t^2} \qquad 0 < x < \pi$$

$$t > 0$$

$$\psi(0,t)=0 \quad , \quad \psi(\pi,t)=0 \qquad t>0$$

$$\psi(x,0)=f(x) \quad , \quad \psi(x,0)=0 \quad 0\le x\le \pi$$

$$F(x)=\begin{cases} x & 0\le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$

$$\frac{\partial^2\psi}{\partial t^2}=V^2\frac{\partial^2\psi}{\partial x^2}\text{..........................}(1)$$

Let:

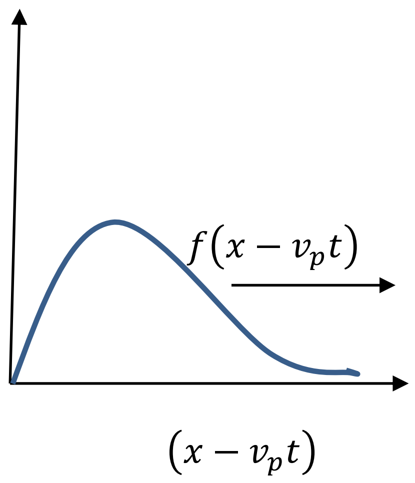
$$\psi\,(x,t)=f(x-v_p t)\text{...............}(2)$$

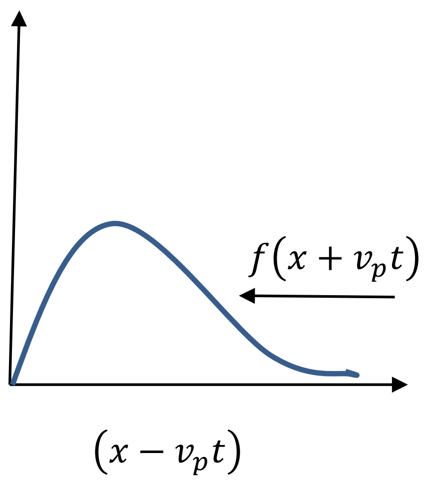
$$\therefore \qquad \frac{\partial^2 f}{\partial t^2}={v_p}^2\frac{\partial^2 f}{\partial x^2}\text{...........................}(3)$$

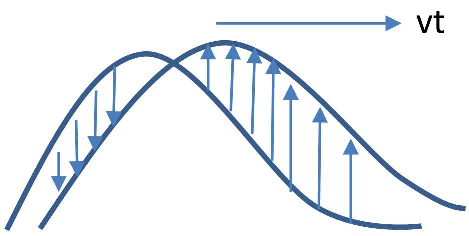
When:

v_p: phase velocity, $f(x-v_pt)$: *wave equation* .

You can also show that any function $f\,(kx\pm\omega t)$ gives the same result, when $(\omega=v_pk)$.



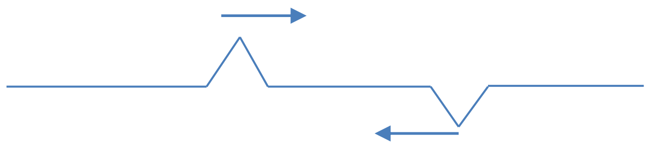


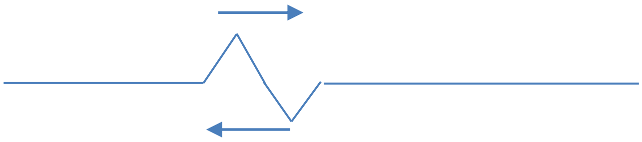


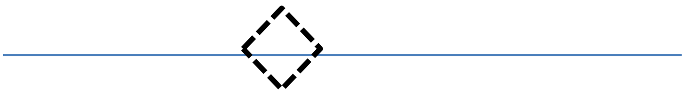
What is moving?

This work for any wave shape.

Example/ Wave equations are linear: this means that a linear combination of solution is a solution.

1- 

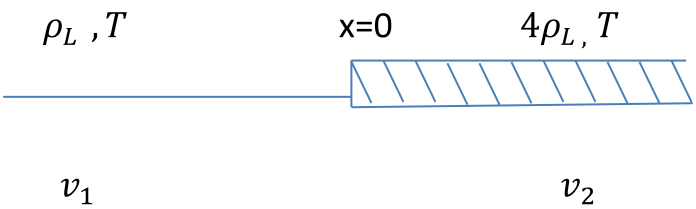
2- 

3- 

This case different from stationary string: the energy stored in the string.

4- 

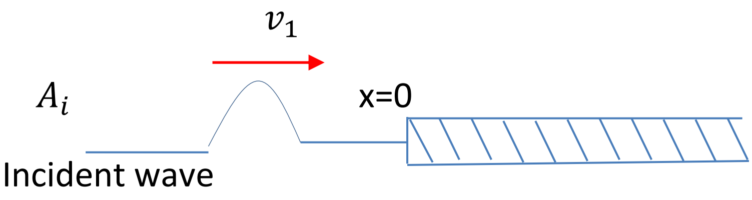
If we have a string with different thicknesses (i.e. different densities (ρ_L) & tensions (T)):

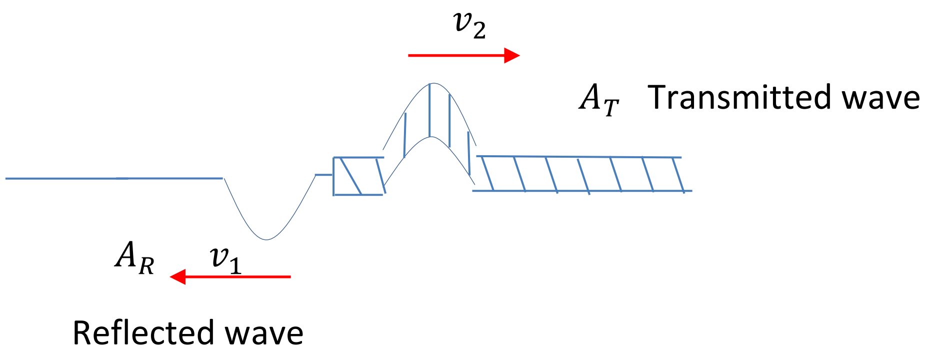


Assuming that the tension (T) is uniform:

$$v_1 = \sqrt{\frac{T}{\rho\, L}} \quad \& \quad v_2 = \sqrt{\frac{T}{4\rho\, L}} = \frac{1}{2} v_1$$

The velocity of a wave v_2 in a denser string$(4\rho)$for example, is slower than $v_1$by half. The wave will moving through this string as:





When a wave moves from one medium to another, the wavelength will be changed but the frequency will be constant.

$$R = \frac{v_2 - v_1}{v_2 + v_1} \qquad \text{Reflected (R)}$$

$$T = \frac{2v_2}{v_2 + v_1} \qquad \text{Transmitted (T)}$$

$v_2 = \frac{v_1}{2}$, & $k \; \alpha \; v^{-1}$

∴ $R = -\frac{1}{3}$, the phase change by (π);

$T = \frac{2}{3}$, no phase change.

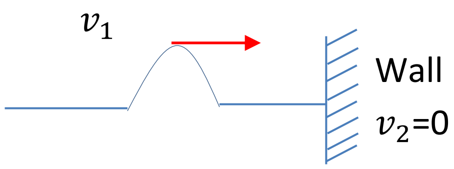
Impedance:

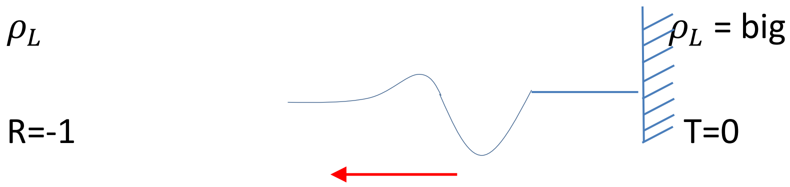
$$Z_1 = \frac{Tension}{v_1} \qquad \& \qquad Z_2 = \frac{Tension}{v_2}$$

The amplitude of the transmitted and reflected wave is determine by the properties of the two media (systems).

Consider two extreme cases:

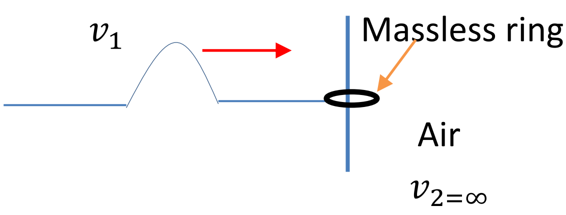
1- String attached a wall:

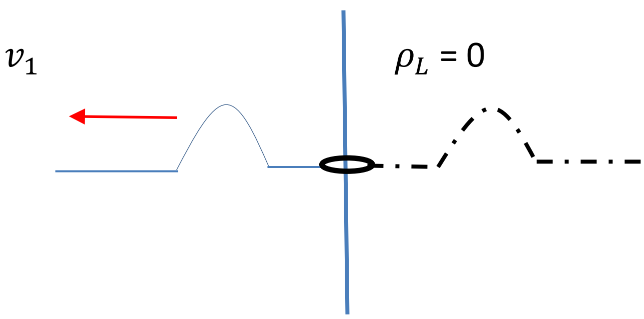




In a sense, the (ρ_L) of the wall is very big, infinite in fact. Therefore, ($v_2 \rightarrow$0), (R=-1) and (T=0). The amplitude changes sign but not magnitude, and there is no transmitted wave.

2- There is air on the other side: the (ρ_L) of the air is zero, therefore, ($v_2 \rightarrow \infty$), (R=1) & (T=2).





EM (Electromagnetic) waves:

Maxwell's equations: differential form

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ Gauss' Law

$\vec{\nabla} \cdot \vec{B} = 0$ Gauss' Law for magnetism

$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$ Faraday's Law

$\vec{\nabla} \times \vec{B} = \mu_0(\vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t})$ Ampere's & Maxwell's Law

: In vacuum

ρ= 0 & J = 0 and we get

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

When in the last two equation see the change in magnetic field generate as electric field and changing electric field product a magnetic field

To solve these equations need to use the identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \left(\vec{\nabla}.\vec{\nabla}\right)\vec{A}$$ When

$$\vec{\nabla}.\vec{\nabla} = \vec{\nabla}^{2}$$ is the Laplace operator

In vacuum

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}\left(\vec{\nabla}.\vec{E}\right) - \vec{\nabla}^{2}\vec{E} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}^{2}\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla}^{2}\vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Where

$$\nabla^{2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

These equations changed the world . Maxwell is the first one who recognized it because of the term he put in. it was represent a wave equation with speed:

$$C=\frac{1}{\sqrt{\mu_o \ \epsilon_o}} = 3 \times 10^8 \mathrm{m/s}$$

Also for B field get:

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

It is very important that the associated magnetic field also satisfies the wave equation. So from Maxwell's equations E crate B and B create E.

Therefore, they cannot exist without each other

1638 Golilo speed of light is very large

1676 Romer speed of light 2.2 $\times 10^8 \mathrm{m/s}$

1724 James Bradley speed of light 3 $\times 10^8 \mathrm{m/s}$

EMW= Oscillation in fields (E & B).

Scalar fields: every position in the space gets a number

Vector Files: Instead of a number or scalar every point gets a vector

$$A_{(x,y,z)} = A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}$$

The electric and magnetic fields are a vector filed.

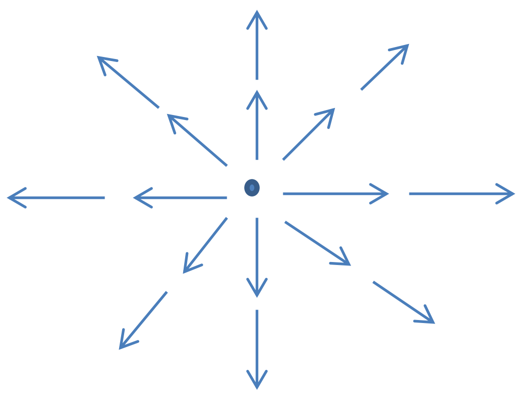
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Divergence: using the definition for above vector:

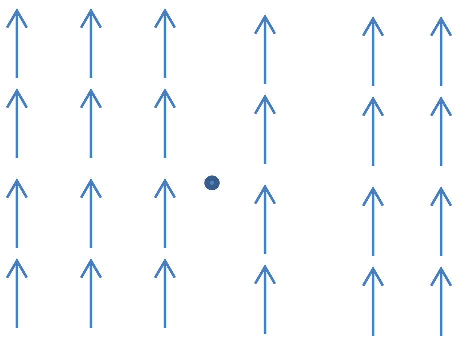
$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\vec{\nabla}.\vec{A} = Div\ A$ Divergence of vector $\vec{A}$

The divergence is a measure of how much the vector spreads out (diverges) from a point:



The divergence of this vector field (B) positive



The divergence of this vector field (B) zero

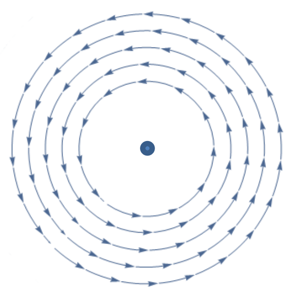
Curl : (rotate)

$$\vec{\nabla} \times \vec{A} = \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{matrix}$$

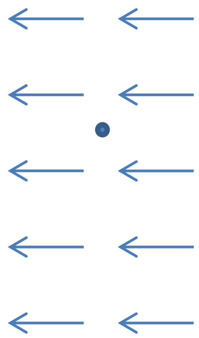
$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

What exactly does Curl mean:

Curl is mean measure of how much the vector $\vec{A}$ (Curl around) a point



This vector field has a large Curl



This vector field has no Curl

Example

Check if a plan wave satisfying wave equations

$$\vec{E} = E_\circ e^{i(kz-wt)} \quad \ldots\ldots\ldots(1) \qquad\qquad take\ only\ real\ part$$

$$\vec{E} = (E_\circ \cos(\mathrm{kz}-\mathrm{wt}), 0, 0) \ldots\ldots(2) \quad and\ find\ \vec{B}$$

Solution:

$$\nabla^2 \mathrm{E} = \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} \ldots\ldots\ldots(3)$$

$$-E_\circ k^2 \cos(\mathrm{kz}-\mathrm{wt}) = -\mu_0\epsilon_0 E_\circ \cos(kz-wt) \ldots\ldots(4)$$

$$\frac{w}{k} = \frac{1}{\sqrt{\mu_o \ \epsilon_o}} = c$$

$$condition\ needed\ to\ satisfy\ the\ wave\ equation$$

Then we can fined $\vec{B}\ field$

$$(\vec{\nabla} \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t} \ldots\ldots(6)$$

R.H.S= $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \left(\frac{\partial E_X}{\partial Z}\hat{\jmath} - \frac{\partial E_z}{\partial y}\hat{k}\right) = -E_\circ \hat{\jmath}\, k \sin(\mathrm{kz}-\mathrm{wt})$

So eq(6) become

$$\frac{\partial \vec{B}}{\partial t} = \hat{\jmath}\, E_\circ\, k \sin(\mathrm{kz}-\mathrm{wt}) \ldots(7)$$

$$\vec{B} = \hat{\jmath}\, kE_\circ \int \sin(kz-wt)\, dt$$

$$\vec{B} = \hat{j}\,\frac{KE_\circ}{W}\cos(\mathrm{kz} - \mathrm{wt}) \ldots (8)$$

$$\vec{B} = \hat{j}\,\frac{E_\circ}{c}\cos(\mathrm{kz} - \mathrm{wt}) \ldots (9)$$

Here can be concluding

1- The $\vec{E}$ field must come with $\vec{B}$ field, the two fields are perpendicular and they are in phase . if $\hat{k}$ is the direction of propagation then

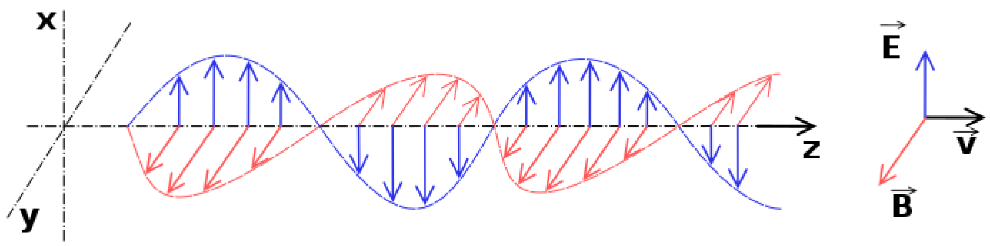
$$\vec{B} = \frac{1}{c} K_u \times \vec{E} \ldots (9)$$

The amplitude of the magnetic field is equal to the amplitude field divided by the speed of light .

2- The EMW is non-dispersive that mean the speed of light (c) is independent of the wave number (k)

$$c = \frac{w}{k}\frac{1}{\sqrt{\mu_o\ \epsilon_o}}$$

3- The direction of propagation of EM is in same direction of $(\vec{\mathrm{E}} \times \vec{B})$.



$$\vec{B} = \frac{1}{c} K_u \times \vec{E}$$

K_u *unit vector in direction of propagation or in* $\vec{k}$ *direction*

Example

$$\vec{K} = \frac{2\pi}{\lambda}\left(\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}}\right), \quad \vec{E_\circ} = -\frac{E_\circ}{\sqrt{2}}\hat{\imath} + \frac{E_\circ}{\sqrt{2}}\hat{\jmath}$$

$$r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{k}.\vec{r} = \frac{2\pi}{\lambda\sqrt{2}}(x + y)$$

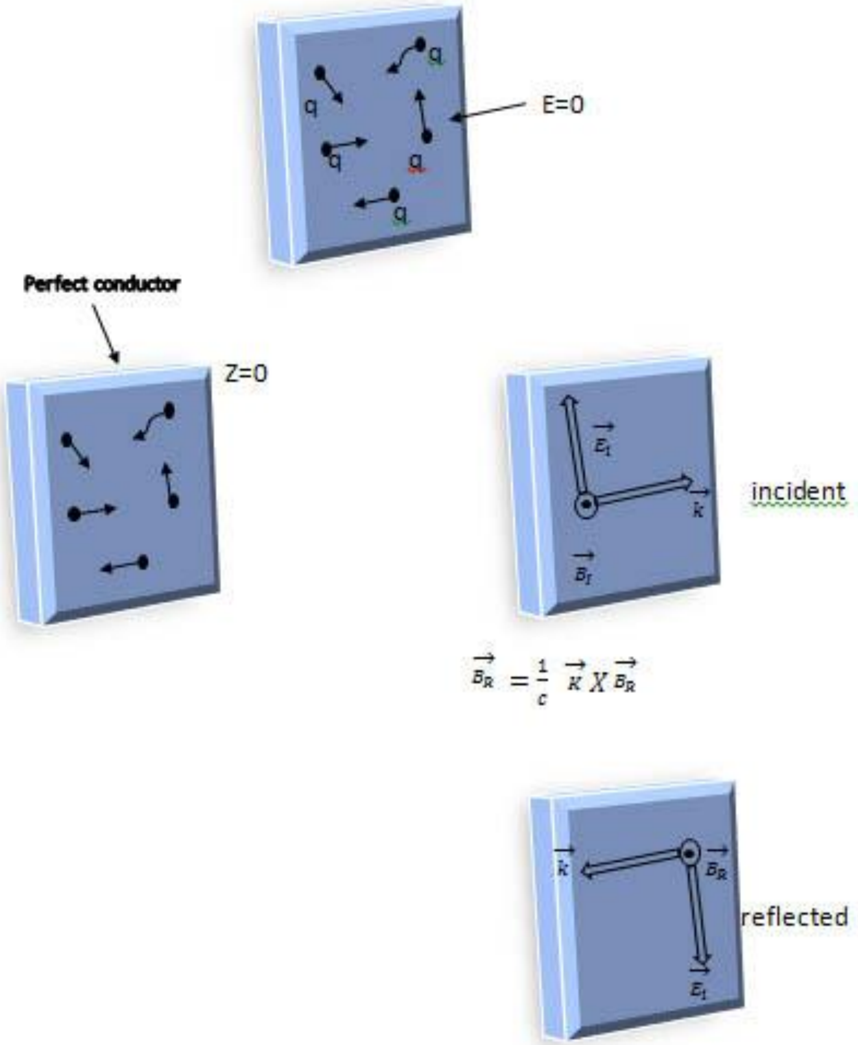
$$\vec{E} = E_\circ\left(-\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}}\right)\cos(\frac{\sqrt{2}\pi}{\lambda}(x + y) - wt)$$

$$\vec{B} = \frac{1}{c}K_u \times \vec{E}$$

$$= \frac{E_\circ}{c}\hat{k}\cos(\frac{\sqrt{2}\pi}{\lambda}(x + y) - wt)$$

If there is no other material , the EMW will travel forever

Now let put a perfect conductor and the EMW incident to this conductor



Incident wave

$$\vec{E}_I = \frac{E_\circ}{2}\cos(kz - wt)\,\hat{\imath}$$

$$\vec{B}_I = \frac{E_\circ}{2c}\cos(kz - wt)\,\hat{\jmath}$$

To satisfy the boundary condition

$$\vec{E}_1 = 0 \; and \quad z = 0$$

We need only a reflected wave:

$$\vec{E}_R = -\frac{E_\circ}{2}\cos(-kz - wt)\,\hat{\imath}$$

$$\vec{B}_R = -\frac{E_\circ}{2c}\cos(-kz - wt)\,\hat{j}$$

$$\vec{E} = \vec{E}_I + \vec{E}_R = \hat{\imath}\frac{E_\circ}{2}\{\cos(kz - wt) - \cos(-kz - wt)\}$$

$$\vec{E} = E_\circ \sin(2wt)\sin(2kz)\hat{\imath}$$

$$\vec{B} = \vec{B}_I + \vec{B}_R = \hat{\jmath}\frac{E_\circ}{2}\{\cos(kz - wt) + \cos(-kz - wt)\}$$

$$\vec{B} = \frac{E_\circ}{c}\cos(2wt)\cos(2kz)$$

Standing wave $\vec{B}$& E field not in phase

$$U_E = \frac{1}{2}\,\varepsilon_\circ E^2 = \frac{\varepsilon_\circ}{2}\,{E_\circ}^2 sin^2 wt + sin^2 kz$$

$$U_B = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{\varepsilon_\circ}{2}\,{E_\circ}^2 cos^2 wt + cos^2 kz$$

Poynting vector : directional energy flux, or the rate of energy transfer per unit area

$$S = \frac{\vec{\mathrm{E}} \times \vec{B}}{\mu_0}$$

$$S = \frac{1}{\mu_0}E_x B_y \hat{k}$$

$$S = \frac{{E_\circ}^2}{c\,\mu_0}\sin wt\; cos\; wt\; sinkz\; coskz$$

$$S = \frac{{E_\circ}^2}{4c\,\mu_0}\sin(2wt)\cos(2kz)$$

This is how a microwave oven work. The EMW are bounced inside the oven .EMW increase the vibration of the molecules in the oven and increase the temperature of the food.

Example

A plane em wave $E = 100\cos(6\times10^8\, t + 4x)$ V/m propagates in a medium. What is the dielectric constant of the medium?

$$E_z = 100\cos(6\times10^8 t + 4x) \text{ (by problem)} \quad (1)$$

$$E_z = A\cos(\omega t + kx) \quad \text{(standard equation)} \quad (2)$$

Comparison of (1) and (2) shows that

$$\omega = 6\times10^8 \text{ and } k = 4$$

$$v = \frac{\omega}{k} = \frac{6\times10^8}{4} = 1.5\times10^8 \text{ m/s}$$

Dielectric constant, $K = \frac{c}{v} = \frac{3\times10^8}{1.5\times10^8} = 2.0$

Example

Show that for a magnetic field $\boldsymbol{B}$ the wave equation has the form $\nabla^2\boldsymbol{B} = \mu_0\varepsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2}$

Maxwell's equations in vacuum are

$$\nabla\cdot\boldsymbol{E} = 0 \quad (1)$$

$$\nabla\cdot\boldsymbol{B} = 0 \quad (2)$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \quad (3)$$

$$\nabla\times\boldsymbol{B} = \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \quad (4)$$

Use the vector identify

$$\nabla\times(\nabla\times\boldsymbol{B}) = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B}$$

$$\therefore \quad \nabla\times\left(\mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right) = -\nabla^2\boldsymbol{B} \quad (\because \nabla\cdot\boldsymbol{B} = 0 \quad \text{by}(2)$$

$$\therefore \quad \mu_0\varepsilon_0\frac{\partial}{\partial t}(\nabla\times\boldsymbol{E}) = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\frac{\partial\boldsymbol{B}}{\partial t} = -\nabla^2\boldsymbol{B}$$

$$\therefore \quad \nabla^2\boldsymbol{B} = \mu_0\varepsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2}$$

Example

Use Maxwell's equation to show that $\nabla.\left(j+\frac{1}{\varepsilon_0}\frac{\partial E}{\partial t}\right)=0.$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \quad \text{(Ampere's law)} \tag{1}$$

Use the vector identity $\boldsymbol{A} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = 0$. Put $\boldsymbol{A} = \nabla$.

$$\therefore \quad \nabla \cdot (\nabla \times \boldsymbol{B}) = 0$$

$$\therefore \quad \nabla \cdot \boldsymbol{j} = 0 \tag{2}$$

More generally, $\nabla \cdot \boldsymbol{j} + \dfrac{\partial \rho}{\partial t} = 0$ (continuity equation) (3)

and $\nabla \cdot \boldsymbol{E} = \varepsilon_0 \rho$ (Gauss' law) (4)

Combining (3) and (4)

$$\nabla \cdot \left(\boldsymbol{j} + \frac{1}{\varepsilon_0}\frac{\partial \boldsymbol{E}}{\partial t}\right) = 0$$

Example

The free-space wave equation for a medium without absorption is

$$\nabla^2 \boldsymbol{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

Show that this equation predicts that electromagnetic waves are propagated with velocity of light given by $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

$$\nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \quad \text{(free-space wave equation)}$$

Compare with the standard three-dimensional wave equation

$$\nabla^2 \Psi = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

$$\therefore \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7})(8.854 \times 10^{-12})}}$$

$$= 2.998 \times 10^8 \text{ m/s} = \text{c}$$

Example

Show that at any point in the electromagnetic field the energy density stored in the electric field is equal to that stored in the magnetic field.

$$u_E = \frac{1}{2}\varepsilon_0 E^2 \quad \text{(energy density in } E\text{-field)} \tag{1}$$

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0} \quad \text{(energy density in } B\text{-field)} \tag{2}$$

The fields for the plane wave are

$$E = E_m \sin(kx - \omega t) \tag{3}$$

$$B = B_m \sin(kx - \omega t) \tag{4}$$

Substituting (3) in (1) and (4) in (2)

$$u_E = \frac{1}{2}\varepsilon_0 E_m^2 \sin^2(kx - \omega t) \tag{5}$$

$$u_B = \frac{1}{2}\frac{B_m^2}{\mu_0} \sin^2(kx - \omega t) \tag{6}$$

Dividing (5) by (6)

$$\frac{u_E}{u_B} = \frac{\varepsilon_0 \mu_0 E_m^2}{B_m^2} \tag{7}$$

But $\varepsilon_0 \mu_0 = \frac{1}{c^2}$ and $E_m = cB_m$

$\therefore \quad \frac{u_E}{u_B} = 1$ or $u_E = u_B$

References

1- 1000 Solved Problems in Classical Physics. Ahmad A. Kamal . 2011.

2- Jackson, John D. (1998). *Classical Electrodynamics (3rd ed.)*. Wiley