Linear Differential Equations

3.1 INTRODUCTION

In this chapter we study linear differential equations. As we have already seen in Chapters 1 and 2, such equations arise naturally as mathematical models for some physical systems. Students familiar with linear algebra can proceed directly to Chapter 4, where more general results are obtained.

We shall concentrate on second-order equations of the form

$$a_0(t)\, y'' + a_1(t)\, y' + a_2(t)\, y = f(t) \tag{3.1}$$

where a_0, a_1, a_2, f are given functions continuous on some interval I; the interval I may be open, closed, or open at one end and closed at the other. We shall see that all the results concerning (3.1) can readily be extended to linear equations of order higher than 2.

Equations of the form (3.1) occur in many applications. For example, the simplest (and least accurate) mathematical model for the simple pendulum is of the form of Eq. (3.1) with $a_0(t) \equiv 1$, $a_1(t) \equiv 0$, $a_2(t) = g/L$, $f(t) \equiv 0$ (see Eq. 2.11, Section 2.2). Many physical problems, such as the motion of a pendulum, a "mass–spring" system, and the oscillations in the shaft of an electric motor, have equations such as (3.1) as their crudest mathematical models. By this we mean that in most instances the mathematical model may well be more complicated; for example, in the case of the simple pendulum, the derivation originally led to a nonlinear differential equation (Eq. 2.8, Section 2.2). In such cases one naturally tries to see whether the relevant equation may be simplified in such a way that the new approximating equation can actually be solved. This process usually involves "linearizing" the equation. In the case of the pendulum equation, we accomplish this by replacing $\sin\theta$ by θ in the equation. Naturally we hope that for "small oscillations" this approximation is good enough to predict the nature of the motion.

It certainly is not obvious at this stage that the linearized equation will be any simpler to handle than the original one. However, experience will show that linear equations are relatively easy to handle, while nonlinear ones usually present serious difficulties.

If one linearizes a problem (for the simple pendulum this means replacing $\sin\theta$ by θ in the equation), the following question arises naturally: How good an approximation does the linearized equation actually produce? For the pendulum, we would like to prove that in some sense the motions of the linear and nonlinear models are close to each other when $|\theta|$ is "small." We can hope to answer such questions only much later (see Section 8.4 for a treatment of the nonlinear simple pendulum with damping). However, the material presented here is an essential first step–before we can ask how good an approximation the linearized equation produces, we must be able to solve this linearized equation.

Before beginning the study of the general theory of Eq. (3.1), we recall that we already know something about this equation. Namely, as an application of the fundamental existence and uniqueness theorem for second-order equations (Theorem 1, Section 2.5; see also Exercise 3, Section 2.5), we can state the following result.

Theorem 1. *Let a_0, a_1, a_2, f be functions continuous on some interval I, and let $a_0(t)\neq 0$ for all t in I. Then for each t_0 in I, there exists one and only one solution $\phi(t)$ of the equation (3.1) satisfying arbitrary prescribed initial conditions $\phi(t_0)=y_0$, $\phi'(t_0)=z_0$. This solution $\phi(t)$ exists on the whole interval I.*

The fact that the solution $\phi(t)$ of the linear equation (3.1) exists on the entire interval I does not follow from Theorem 1, Section 2.5, but can be proved separately (see, for example, Exercise 1, Section 8.5). In this chapter, we shall assume the validity of Theorem 1 as stated. We may formulate this in another way. For a linear second-order differential equation a solution with a given initial displacement and slope exists and is unique for as long as the coefficients are continuous and the coefficient of the leading term ($a_0(t)$ in (3.1)) is not zero.

Example 1. Consider the differential equation $ty''+(\cos t)y'+[1-1/(t+1)]y=2t$. Discuss existence and uniqueness of solutions.

Here $a_0(t)=t$, $a_1(t)=\cos t$, $a_2(t)=1-1/(t+1)$, $f(t)=2t$ are continuous for all t except $a_2(t)$, which is discontinuous at $t=-1$; also $a_0(0)=0$. Thus we must distinguish three cases for the initial time t_0: *Case* (*i*): $t_0<-1$; *Case* (*ii*): $-1<t_0<0$; *Case* (*iii*): $t_0>0$. We do not take $t_0=0$ or $t_0=-1$ (why?). In case (i), by Theorem 1, given any $t_0<-1$, there exists one and only one solution ϕ of the given equation satisfying the initial conditions $\phi(t_0)=y_0$, $\phi'(t_0)=z_0$, where y_0, z_0 are arbitrary given real numbers; this solution ϕ exists on the interval $-\infty<t<-1$ by the last statement in Theorem 1.

Exercises

1. Discuss in a similar way the existence and uniqueness problem for cases (ii) and (iii) of the equation in Example 1.
2. Discuss the existence and uniqueness problem for real solutions of the equation
$$(1+t)\,y''+2ty'+(\log|t|)\,y=\cos t.$$
3. Do the same for the equation
$$a_0y''+a_1y'+a_2y=f(t)$$
where a_0, a_1, a_2 are *constants* and $f(t)$ is continuous on $-\infty<t<\infty$.

Example 2. Consider

$$a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=0, \tag{3.2}$$

where a_0, a_1, a_2 are continuous on some interval I and $a_0(t)\neq 0$ on I. Show that $\phi(t)\equiv 0$ is the only solution satisfying the initial conditions $\phi(t_0)=0$, $\phi'(t_0)=0$, where t_0 is in I.

It is readily verified that the function ϕ defined by $\phi(t)=0$ for all t in I is a solution of this initial value problem. Therefore, by Theorem 1 (here $f(t)\equiv 0$), $\phi(t)\equiv 0$ is the *only solution* on Eq. (3.2) on I satisfying the initial conditions $\phi(t_0)=0$, $\phi'(t_0)=0$ for any t_0 in I.

Exercises

4. Show that, if solutions ϕ of Eq. (3.2) are represented as curves in the (t, y) plane, *no* solution of (3.2) except $\phi(t)\equiv 0$ can be tangent to the t axis at any point of I. [*Hint*: Study Example 2.]
5. For each of the following differential equations, determine the largest intervals on which a unique solution is certain to exist by application of Theorem 1. In each case, it is assumed that you are given initial conditions of the form $\phi(t_0)=y_0$, $\phi'(t_0)=z_0$ with t_0 arbitrary. Note that the interval to be determined may depend on the choice of t_0.

a) $ty''+y=t^2$ b) $t^2(t-3)\,y''+y''=0$

c) $y''+\sqrt{t}\,y=0.$ d) $(1+t^2)\,y''-y'+ty=\cos t$

e) $y''-(\sin t)\,y'+y=t^3$ f) $y''-(\log|t|)\,y=0$

For the linear differential equation of order n

$$a_0(t)\,y^{(n)}+a_1(t)\,y^{(n-1)}+\cdots+a_{n-1}(t)\,y'+a_n(t)\,y=f(t), \tag{3.3}$$

the following analog of Theorem 1, partly a consequence of Theorem 2, Section 2.5, is valid.

Theorem 2. *Let $a_0, a_1, \ldots, a_n, f$ be continuous functions on some interval I and suppose $a_0(t)\neq 0$ for all t in I. Then, for each t_0 in I, there exists one and only one solution $\phi(t)$ of Eq. (3.3) which satisfies arbitrary prescribed initial conditions*

$$\phi(t_0)=\alpha_1, \quad \phi'(t_0)=\alpha_2, \quad \phi''(t_0)=\alpha_3, \ldots, \phi^{(n-1)}(t_0)=\alpha_n.$$

The solution $\phi(t)$ exists on the entire interval I.

As for Theorem 1, the fact that solutions exist on the whole interval I does not follow from Theorem 2, Section 2.5 (see Exercise 2, Section 8.5).

Exercise

6. Apply Theorem 2 to determine the largest intervals on which the existence of a unique solution $\phi(t)$ is assumed when initial conditions of the form

$$\phi(t_0)=\alpha_1, \quad \phi'(t_0)=\alpha_2, \ldots, \phi^{(n-1)}(t_0)=\alpha_n$$

are given; distinguish different values of t_0 if necessary.

a) $y'''+(\cos t)\,y'+(1-t^2)\,y=e^t$ b) $y'''+(\cos t)\,y'+(1-t^2)\,y=\tan t$

c) $ty^{(4)}+y=e^{-t}+\cos t$ d) $ty^{(4)}+y=\sec t$

3.2 LINEARITY

To develop the theory of linear differential equations, such as (3.1), it is convenient to introduce the *operator* L defined by the relation

$$L(y)(t)=a_0(t)\,y''(t)+a_1(t)\,y'(t)+a_2(t)\,y(t) \tag{3.4}$$

which we denote briefly by $L(y)$, where $L(y)=a_0y''+a_1y'+a_2y$. Here we think of $L(y)(t)$ as the value of the function $L(y)$ at the point t. Noticing that $L(y)$ is precisely the left-hand side of Eq. (3.1), we may write the equation simply as

$$L(y)=f \tag{3.5}$$

where it is understood that all functions are functions of t.

An operator is, roughly speaking, a function applied to functions. In the present case, the operator L is a rule which assigns to each twice differentiable function y on some interval I the function $L(y)$, where $L(y)(t)=a_0(t)\,y''(t)+a_1(t)\,y'(t)+a_2(t)\,y(t)$.

The operator L is a particular example of a class of operators called linear operators: *An operator T defined on a collection S of functions is said to be linear if and only if for any two functions y_1 and y_2 in the collection S and for any constants c_1 and c_2 one has*

$$T(c_1y_1+c_2y_2)=c_1T(y_1)+c_2T(y_2).$$

It is easy to verify that our operator L defined by (3.4) is linear. To see this, let S be the collection of twice differentiable functions defined on the interval I. Then if y_1 and y_2 are any two functions in S and c_1 and c_2 are any two constants, $L(c_1y_1+c_2y_2)=a_0(c_1y_1+c_2y_2)''+a_1(c_1y_1+c_2y_2)'+a_2(c_1y_1$

$+c_2y_2)=c_1L(y_1)+c_2L(y_2)$ by elementary facts about differentiation (which ones?).

Exercises

1. Show that the operator T defined by $T(y)(t)=\int_a^t y(s)\,ds$, for any function y continuous on $a\leq t\leq b$, is a linear operator.
2. Give other examples of linear operators.
3. Show that the operator T defined by $T(y)=(y')^2$, for any function y differentiable one some interval I, is not linear.

We shall need some more terminology before proceeding to the theory of linear differential equations. If the function $f\not\equiv 0$ on I, we say that Eq. (3.5) is *nonhomogeneous* (with nonhomogeneous term f). With every nonhomogeneous linear differential equation of the form (3.5) we associate the *homogeneous* (or reduced) linear differential equation $L(y)=0$ obtained from (3.5) by replacing f by the zero function.

We now give two basic properties of solutions of linear differential equations; these are immediate consequences of the linearity of the operator L.

i) *If ϕ_1 and ϕ_2 are any two solutions of the homogeneous linear differential equation $L(y)=0$ on some interval I, then for any constants c_1 and c_2 the function $c_1\phi_1+c_2\phi_2$* (called a *linear combination of ϕ_1 and ϕ_2*) *is also a solution of $L(y)=0$ on I.*

To see this we merely compute: $L(c_1\phi_1+c_2\phi_2)=c_1L(\phi_1)+c_2L(\phi_2)$, by the linearity of L. Since ϕ_1 and ϕ_2 are solutions of $L(y)=0$ on I, $L(\phi_1)=L(\phi_2)=0$ for every t on I, and therefore $L(c_1\phi_1+c_2\phi_2)=0$. Thus $c_1\phi_1+c_2\phi_2$ is a solution of $L(y)=0$. ∎

Exercise

4. Use mathematical induction and the above result to establish the analog of property (i) for m solutions $\phi_1(t),\ldots,\phi_m(t)$ of $L(y)=0$; that is, show that if $\phi_1,\phi_2,\ldots,\phi_m$ are m solutions of $L(y)=0$ on I and if $c_1,c_2,\ldots,c_m$ are any constants, then $c_1\phi_1+c_2\phi_2+\cdots+c_m\phi_m$ is a solution of $L(y)=0$ on I.

This result is usually expressed by saying that any linear combination of solutions of $L(y)=0$ is again a solution of $L(y)=0$. It is sometimes called *the principle of superposition* of solutions. Our object in the next section will be to show that the problem of solving the equation $L(y)=0$ can be reduced to the problem of finding certain special solutions of $L(y)=0$ and obtaining all other solutions as linear combinations of these special solutions.

Another important consequence of the linearity of the operator L is the following.

ii) *If ϕ and ψ are any two solutions of the nonhomogeneous linear differential equation $L(y)=f$ on some interval I, then $\phi-\psi$ is a solution of the corresponding homogeneous equation $L(y)=0$.*

To see this, we merely compute $L(\phi-\psi)$. By the linearity of L we have $L(\phi-\psi)=L(\phi)-L(\psi)$, for t in I. But ϕ and ψ are solutions of $L(y)=f$ on I. Therefore $L(\phi-\psi)=f-f=0$ for t in I, which proves the result. ∎

This result shows that it is only necessary to find one solution of the equation $L(y)=f$, *provided* that one knows all solutions of $L(y)=0$. This is because every other solution of the nonhomogeneous equation (3.5) differs from the known one by some solution of the homogeneous equation $L(y)=0$.

Exercises

5. Given that u is a solution of $L(y)=0$ and v is a solution of $L(y)=f$ on some interval I, show that $u+v$ is a solution of $L(y)=f$ on I.
6. Suppose f can be written as the sum of m functions $f_1,\dots,f_m$; that is, $f(t)=f_1(t)+f_2(t)+\cdots+f_m(t)$, for t on some interval I. Suppose that u_1 is a solution of the linear equation $L(y)=f_1$, u_2 is a solution of the linear equation $L(y)=f_2$, and in general u_i is a solution of the linear equation $L(y)=f_i$ on I for $i=1,\dots,m$. Show that the function $u=u_1+u_2+\cdots+u_m$ is a solution of $L(y)=f$ on I. (This result, also called *the principle of superposition*, enables us to decompose the problem of solving $L(y)=f$ into simpler problems in certain cases.)

Before closing this section we repeat that the only property of the operator L is used above is linearity. Therefore our results are much more general than appears to be the case. In particular, if we define the linear differential operator L_n of order n by the relation

$$L_n(y)(t)=a_0(t)\,y^{(n)}(t)+a_1(t)\,y^{(n-1)}(t)+\cdots+a_{n-1}(t)\,y'(t)+a_n(t)\,y(t)$$

where y is any function which is n times differentiable on some interval I, and the functions a_j $(j=0,1,\dots,n)$ are continuous on I, $a_0(t)\neq 0$ on I, then all results stated in Section 3.2 hold.

Exercise

7. Formulate and verify the analogs of the linearity properties (i) and (ii) for the equation $L_n(y)=f$ for $n=1,3,4$, and n an arbitrary positive integer.

3.3 LINEAR HOMOGENEOUS EQUATIONS

In this section we go far beyond the result established above, that any linear combination of solutions of the linear homogeneous differential equation $L(y)=0$ is again a solution of $L(y)=0$. We will show that *every* solution of $L(y)=0$ is a linear combination of certain special solutions. Then in Section 3.7 we will show how to use the special solutions to find every solution of the nonhomogeneous equation $L(y)=f$, using the linearity property ii) established in the previous section.

Before we can do this we need the important concept of linear dependence.

Definition. *We say that m functions* $g_1, g_2, \ldots, g_m$ *are linearly dependent on an interval I if and only if there exist constants* $b_1, b_2, \ldots, b_m$, *not all zero, such that*

$$b_1 g_1(t) + b_2 g_2(t) + \cdots + b_m g_m(t) = 0$$

for every t on I. We say further that the m functions are linearly independent on I if they are not linearly dependent on I.

Example 1. Show that the functions $\sin^2 t$, $\cos^2 t$, 1 are linearly dependent on any interval.

Since $\sin^2 t + \cos^2 t - 1 \equiv 0$ for every t, we merely put $g_1(t) = \sin^2 t$, $g_2(t) = \cos^2 t$, $g_3(t) = 1$, $b_1 = b_2 = 1$, $b_3 = -1$ in the above definition. This proves the linear dependence of the given functions.

Example 2. Show that the functions $e^{r_1 t}$, $e^{r_2 t}$, where r_1, r_2 are real constants, are linearly independent on any interval I provided that $r_1 \neq r_2$.

To see this, we suppose that there exist constants b_1, b_2 such that $b_1 e^{r_1 t} + b_2 e^{r_2 t} = 0$ for all t in I. Multiplying by $e^{-r_1 t}$ we obtain $b_1 + b_2 e^{(r_2 - r_1)t} = 0$ for all t in I, and differentiating both sides of this equation with respect to t, we obtain $b_2(r_2 - r_1)e^{(r_2 - r_1)t} = 0$ for all t in I. Since $r_1 \neq r_2$ and $e^{(r_2 - r_1)t}$ is never zero, this implies that b_2 must be zero. However, then $b_1 e^{r_1 t} + b_2 e^{r_2 t} = 0$ for all t in I, implies $b_1 e^{r_1 t} = 0$ for all t in I, and hence b_1 must also be zero. Since b_1 and b_2 are both zero, $e^{r_1 t}$ and $e^{r_2 t}$ must be linearly independent.

Exercises

1. Establish the linear independence of the following sets of functions on the intervals indicated.
   a) $\sin t$, $\cos t$ on any interval I.
   b) $e^{r_1 t}$, $e^{r_2 t}$, $e^{r_3 t}$ on any interval I if r_1, r_2, r_3 are all different.
   c) $e^{r_1 t}$, $te^{r_1 t}$ on any interval I.
   d) $1, t, t^2, t^3$ on any interval I.
   e) t^2, $t|t|$ on $-1 \leq t \leq 1$ but *not* on $0 \leq t \leq 1$.
   f) The functions $f_1(t)$, $f_2(t)$ on $-1 < t < 1$, where

$$f_1(t) = \sum_{n=0}^{\infty} t^{2n}, \quad f_2(t) = \sum_{n=0}^{\infty} (-1)^n t^{2n+1}.$$

2. Prove that the functions f, g are linearly dependent on I if and only if there exists a constant c such that either $f(t) = cg(t)$ or $g(t) = cf(t)$ for every t in I.
3. Decide which of the following sets of functions are linearly dependent and which are linearly independent on the given interval. Justify your answer in each case.

a) $\phi_1(t)=e^t,\quad \phi_2(t)=e^{t+1},\quad -\infty<t<\infty$

b) $\phi_1(t)=e^{2t},\quad \phi_2(t)=e^t,\quad -\infty<t<\infty$

c) $\phi_1(t)=\sqrt{t},\quad \phi_2(t)=t,\quad 0<t<\infty$

d) $\phi_1(t)=1,\quad \phi_2(t)=e^t,\quad \phi_3(t)=e^{-t},\quad -\infty<t<\infty$

e) $\phi_1(t)=t^2,\quad \phi_2(t)=t^2\sin t,\quad -1\le t\le 1$

f) $\phi_1(t)=1,\quad \phi_2(t)=\begin{cases}0 (t<0)\\ t\ (t\ge 0)\end{cases},\quad \phi_3(t)=\begin{cases}0\ (t<0)\\ t^2(t\ge 0)\end{cases}\quad -\infty<t<-1$

g) $\phi_1(t)=1,\quad \phi_2(t)=\begin{cases}0 (t<0)\\ t\ (t\ge 0)\end{cases},\quad \phi_3(t)=\begin{cases}0\ (t<0)\\ t^2(t\ge 0)\end{cases}\quad -\infty<t<\infty$

h) $\phi_1(t)=t^2,\quad \phi_2(t)=t^4,\quad \phi_3(t)=t^6,\quad \phi_4(t)=t^{10},\quad -1\le t\le 1$

More generally, we have the following result, which will be useful on several occasions later.

Lemma 1. *The n functions*

$$\begin{array}{llll}
e^{r_1t}, & te^{r_1t}, & \ldots, & t^{k_1-1}e^{r_1t},\\
e^{r_2t}, & te^{r_2t}, & \ldots, & t^{k_2-1}e^{r_2t},\\
\cdot & \cdot & & \cdot\\
\cdot & \cdot & & \cdot\\
\cdot & \cdot & & \cdot\\
e^{r_it}, & te^{r_it}, & \ldots, & t^{k_i-1}e^{r_it}\\
\cdot & \cdot & & \cdot\\
\cdot & \cdot & & \cdot\\
\cdot & \cdot & & \cdot\\
e^{r_st}, & te^{r_st}, & \ldots, & t^{k_s-1}e^{r_st},
\end{array}$$

where $k_1+k_2+\ldots+k_s=n$ and where $r_1, r_2, \ldots, r_s$ are distinct numbers, are linearly independent on every interval I.

Since the proof of this theorem is technically rather complicated, the reader is advised not to get involved in the details. He should be sure he understands the statement of the Lemma. Note that Example 2 is a special case with $k_1=k_2=1$, $n=2$.

Proof. The proof is an extension of the argument used in Example 2 above. Suppose the n functions are linearly dependent on some interval I. Then there exist n constants a_{ij}, $i=1, 2, \ldots, s$, $j=0, 1, \ldots, k_i-1$, not all zero, such that

$$a_{10}e^{r_1t}+a_{11}te^{r_1t}+\cdots+a_{1,k_1-1}t^{k_1-1}e^{r_1t}+a_{20}e^{r_2t}+a_{21}te^{r_2t}+\cdots$$
$$+a_{2,k_2-1}t^{k_2-1}e^{r_2t}+\cdots+a_{s0}e^{r_st}+a_{s1}te^{r_st}+\cdots+a_{s,k_s-1}t^{k_s-1}e^{r_st}=0$$

or, more compactly,

$$\sum_{i=1}^{s}\left(a_{i0}e^{r_it}+a_{i1}te^{r_it}+\cdots+a_{i,k_i-1}t^{k_i-1}e^{r_it}\right)=0$$

for all t in I. We may define the polynomials

$$P_i(t) = a_{i0} + a_{i1}t + \cdots + a_{i,k_i-1}t^{k_i-1} \qquad (i = 1, \ldots, s)$$

to write this condition in the form

$$P_1(t)\,e^{r_1 t} + P_2(t)\,e^{r_2 t} + \cdots + P_s(t)\,e^{r_s t} = 0 \tag{3.6}$$

for all t in I. Since, by assumption, the constants a_{ij} are not all zero, at least one of the polynomials $P(t)$ is not identically zero. It is convenient to assume that $P_s(t) \not\equiv 0$; we can always arrange this by a suitable labeling of the numbers $r_1, r_2, \ldots, r_s$. Now we divide Eq. (3.6) by $e^{r_1 t}$ and differentiate at most k_1 times until the first term drops out. Note that all terms in (3.6) can be differentiated as often as we wish. Then we have an equation of the form

$$Q_2(t)\,e^{(r_2-r_1)t} + Q_3(t)\,e^{(r_3-r_1)t} + \cdots + Q_s(t)\,e^{(r_s-r_1)t} = 0 \tag{3.7}$$

for every t in I. The term $Q_i(t)\,e^{(r_i-r_1)t}$ in (3.7) is obtained by differentiating $P_i(t)\,e^{(r_i-r_1)t}$ $(i = 2, \ldots, s)$, as often as necessary to remove the first term $P_1(t)$. Note that differentiation of a polynomial multiplied by an exponential gives a polynomial *of the same degree* multiplied by the same exponential (think of the rule for differentiation of products). Thus the polynomial Q_s in (3.7) has the same degree as P_s, and does not vanish identically. We continue this procedure, dividing by the exponential in the first term and then differentiating often enough to remove the first term, until we are left with only one term. Then we have an equation of the form

$$R_s(t)\,e^{(r_s-r_{s-1})t} \equiv 0$$

in which the polynomial R_s has the same degree as P_s, and does not vanish identically. However, the exponential term in this equation does not vanish, and we have a contradiction. This shows that all the constants a_{ij} must be zero, and therefore that the n given functions are linearly independent on I. ∎

Exercise

4. To which of the sets of functions in Exercises 1 and 3 could you apply Lemma 1 to deduce either linear dependence or linear independence?

The above discussion of linear dependence and independence of functions has not been, up to this point, related to the differential equation $L(y) = 0$. Before continuing, review Theorem 1 and the notion of linearity as given in Section 3.2. Using these, we now establish one of the key results of the theory of linear differential equations.

Theorem 1. *Let a_0, a_1, a_2 be functions continuous on some interval I and let $a_0(t) \neq 0$ for all t on I. Then the differential equation*

$$L(y) = a_0(t)\, y'' + a_1(t)\, y' + a_2(t)\, y = 0$$

has two linearly independent solutions ϕ_1, ϕ_2 on I. Moreover, if ϕ is any solution of $L(y)=0$ on I, then it is possible to find a unique pair of constants c_1, c_2 such that for every t on I

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t).$$

Proof. Let t_0 be any point of the interval I. By Theorem 1, Section 3.1, there exists a unique solution ϕ_1 on I of $L(y)=0$ satisfying the special initial conditions $\phi_1(t_0)=1$, $\phi_1'(t_0)=0$. Similarly, there exists on I a unique solution ϕ_2 of $L(y)=0$ such that $\phi_2(t_0)=0$, $\phi_2'(t_0)=1$. We select these particular solutions because it will be easy to prove that they are linearly independent on I. You will see later, after studying the proof, that many other choices are possible (see Exercises 7 and 8).

We claim first that the solutions ϕ_1 and ϕ_2 are linearly independent on I. Suppose there exist constants b_1, b_2 such that

$$b_1\phi_1(t) + b_2\phi_2(t) = 0 \tag{3.8}$$

for every t on I. Since ϕ_1, ϕ_2 are solutions of $L(y)=0$ on I, they are differentiable on I and hence from (3.8) we have also

$$b_1\phi_1'(t) + b_2\phi_2'(t) = 0 \tag{3.9}$$

for every t on I. In particular, putting $t=t_0$ in (3.8) and (3.9), we obtain respectively from the chosen initial conditions

$$b_1 \cdot 1 + b_2 \cdot 0 = 0, \qquad b_1 \cdot 0 + b_2 \cdot 1 = 0,$$

and we therefore conclude that $b_1 = b_2 = 0$, which shows that the solutions ϕ_1, ϕ_2 cannot be linearly dependent on I and therefore this proves their linear independence on I.

To complete the proof of the theorem, let ϕ be any solution of $L(y)=0$ on I and calculate $\phi(t_0)=\alpha$, $\phi'(t_0)=\beta$. (That is, we evaluate $\phi(t)$ and $\phi'(t)$ at $t=t_0$ and call the values at t_0, α and β, respectively.) If there are to exist constants c_1 and c_2 such that $\phi(t)=c_1\phi_1(t)+c_2\phi_2(t)$ for all t in I, this relation must hold in particular at t_0, and we must have

$$\alpha = \phi(t_0) = c_1\phi_1(t_0) + c_2\phi_2(t_0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1,$$

$$\beta = \phi'(t_0) = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = c_1 \cdot 0 + c_2 \cdot 1 = c_2.$$

Define the function ψ by the relation $\psi(t) = \alpha\phi_1(t) + \beta\phi_2(t)$ for t in I. Clearly (by the linearity property (i), Section 3.2), ψ is a solution of $L(y)=0$ on I; moreover,

$$\psi(t_0)=\alpha\phi_1(t_0)+\beta\phi_2(t_0)=\alpha\cdot 1+\beta\cdot 0=\alpha,$$
$$\psi'(t_0)=\alpha\phi_1'(t_0)+\beta\phi_2'(t_0)=\alpha\cdot 0+\beta\cdot 1=\beta.$$

Therefore ϕ and ψ are both solutions of $L(y)=0$ on I which satisfy the same pair of initial conditions at t_0. Since, by Theorem 1, Section 3.1, there is only one such solution, we conclude that $\phi(t)=\psi(t)=\alpha\phi_1(t)+\beta\phi_2(t)$ on I, which completes the proof.

Exercises

5. Why are the constants c_1, c_2 in the statement of the theorem unique?
6. Carry out the proof of Theorem 1 by using the solutions ψ_1 and ψ_2 of (3.5) on I satisfying the initial conditions $\psi_1(t_0)=2$, $\psi_1'(t_0)=-1$ and $\psi_2(t_0)=-1$, $\psi_2'(t_0)=1$ in place of the solutions ϕ_1 and ϕ_2. [*Hint*: Begin by showing that the solutions ψ_1, ψ_2 of (3.5) are linearly independent on I.]
7. Let w_1 and w_2 be solutions of $L(y)=0$ on I satisfying the initial conditions

$$w_1(t_0)=\alpha,\quad w_1'(t_0)=\beta;\quad w_2(t_0)=\gamma,\quad w_2'(t_0)=\delta$$

respectively. Under what conditions on $\alpha, \beta, \gamma, \delta$ will the solutions w_1, w_2 be linearly independent on I?

8. Assuming the condition found in Exercise 7 to be satisfied, use the solutions w_1 and w_2 to complete the proof of Theorem 1.

Example 3. Find that solution ϕ of $y''+y=0$ such that $\phi(0)=1$, $\phi'(0)=-1$, using the fact that $\cos t$ and $\sin t$ are both solutions.

It is easily shown that $\cos t$ and $\sin t$ are linearly independent solutions of $y''+y=0$ on any interval I (see Exercise 1a).) To find the desired solution we apply Theorem 1, letting $\phi_1(t)=\cos t$, $\phi_2(t)=\sin t$, and observing that $\phi_1(0)=1$, $\phi_1'(0)=0$, $\phi_2(0)=0$, $\phi_2'(0)=1$ as in the above proof. By Theorem 1 we know that there exist unique constants c_1, c_2 such that $\phi(t)=c_1\cos t+c_2\sin t$; as we saw in the proof we may determine c_1 and c_2 by imposing the initial conditions. Thus we obtain

$$\phi(0)=1=c_1\cdot 1+c_2\cdot 0$$
$$\phi'(0)=-1=-c_1\cdot 0+c_2\cdot 1$$

Therefore $c_1=1$, $c_2=-1$ and the desired solution ϕ is $\phi(t)=\cos t-\sin t$.

Exercise

9. State and prove a theorem analogous to Theorem 1 for the linear third-order differential equation

$$L_3(y)=a_0(t)\,y'''+a_1(t)\,y''=a_2(t)\,y'+a_3(t)\,y=0,$$

where a_0, a_1, a_2, a_3 are continuous on some interval I and $a_0(t)\neq 0$ on I. [*Hint*: For any t_0 on I let ϕ_1 be that solution of $L_3(y)=0$ for which $\phi_1(t_0)=1$, $\phi_1'(t_0)=0$, $\phi_1''(t_0)=0$, let ϕ_2 be that solution of $L_3(y)=0$ for which $\phi_2(t_0)=0$,

$\phi_2'(t_0)=1$, $\phi_2''(t_0)=0$. let ϕ_3 be that solution of $L_3(y)=0$ for which $\phi_3(t_0)=0$, $\phi_3'(t_0)=0$, $\phi_3''(t_0)=1$ and now proceed as in the proof of Theorem 1.]

For the general case of linear differential equations of order n,

$$L_n(y)=a_0(t)\,y^{(n)}+a_1(t)\,y^{(n-1)}+\cdots+a_{n-1}(t)\,y'+a_n(t)\,y=0,$$

Theorem 1 has the following analog:

Theorem 2. *Let $a_0, a_1, \ldots, a_n$ be continuous functions on some interval I, and suppose $a_0(t)\neq 0$ on I. Then the differential equation $L_n(y)=0$ has n linearly independent solutions $\phi_1, \phi_2, \ldots, \phi_n$ on I. Moreover, if ϕ is a solution of $L_n(y)=0$ on I, then there exist uniquely determined constants $c_1, c_2, \ldots, c_n$ such that*

$$\phi(t)=c_1\phi_1(t)+c_2\phi_2(t)+\cdots+c_n\phi_n(t)$$

for every t in I.

Exercise

10. Prove Theorem 2.

In practice it is undesirable to restrict ourselves to solutions ϕ_1, ϕ_2 which satisfy special initial conditions such as $\phi_1(t_0)=1$, $\phi_1'(t_0)=0$, $\phi_2(t_0)=0$, $\phi_2'(t_0)=1$ at some t_0 in I. We shall show shortly that instead of the special solutions ϕ_1, ϕ_2 used, *any* two linearly independent solutions of $L(y)=0$ on I will serve the purpose just as well. To see this we can use the result of Exercises 7 and 8. Alternatively, it is convenient to introduce the concept of the *Wronskian*, which, as we shall see, also serves another purpose.

Definition. *Let f_1, f_2, be any two differentiable functions on some interval I. Then the determinant*

$$W(f_1, f_2)=\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}=f_1f_2'-f_1'f_2$$

is called the Wronskian of f_1 and f_2. Its value at any t in I will be denoted by $W(f_1, f_2)(t)$. More generally, if $f_1, \ldots, f_n$ are n functions which are $n-1$ times differentiable on I, then the nth-*order determinant*

$$W(f_1, \cdots, f_n)=\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of $f_1, \ldots, f_n$.

Exercise

11. Evaluate the Wronskian of the following functions

a) $f_1(t)=\sin t$, $f_2(t)=\cos t$, $(-\infty<t<\infty)$
b) $f_1(t)=e^t$, $f_2(t)=e^{-t}$, $(-\infty<t<\infty)$
c) $f_1(t)=t^2$, $f_2(t)=t|t|$, $(-\infty<t<\infty)$
d) $f_1(t)=1$, $f_2(t)=1$, $f_3(t)=t^2$, $(-\infty<t<\infty)$

The Wronskian of two solutions of $L(y)=0$ on I provides us with the following simple test of their linear independence.

Theorem 3. *Let a_0, a_1, a_2 be given functions continuous on some interval I, and let $a_0(t)\neq 0$ for all t on I. Then two solutions ϕ_1, ϕ_2 of*

$$L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=0$$

are linearly independent on I if and only if $W(\phi_1, \phi_2)(t)\neq 0$ for all t on I.

Before proving this result we give an illustration.

Example 4. Show that $\cos t$ and $\sin t$ are linearly independent solutions of $y''+y=0$ for $-\infty<t<\infty$.

The functions $\phi_1(t)=\cos t$, $\phi_2(t)=\sin t$ are solutions of $y''+y=0$ on $-\infty<t<\infty$. To test their linear independence we compute their Wronskian

$$W(\cos t, \sin t)=\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}\equiv 1, \quad -\infty<t<\infty.$$

Therefore, by Theorem 3, $\phi_1(t)=\cos t$, $\phi_2(t)=\sin t$ are linearly independent solutions of $y''+y=0$ on $-\infty<t<\infty$. Of course, we already know this result from having applied the definition of linear independence directly. However, when dealing with solutions of a linear homogeneous equation $L(y)=0$, the theorem is often easier to use than the definition.

Warning. Do not apply Theorem 3 when the functions being tested for linear independence are *not known* to be solutions of a linear homogeneous equation $L(y)=0$. To see why, consider the functions $f_1(t)=t^2$, $f_2(t)=t|t|$ and take for I the interval $-1\leq t\leq 1$. Then as we saw in Exercise 11 (c), the functions f_1, f_2 are linearly independent on I and yet $W(f_1, f_2)(t)=0$ for every t on $-1\leq t\leq 1$.

Proof of Theorem 3. The proof consists of two parts. Suppose first that the solutions $\phi_1(t)$, $\phi_2(t)$ of $L(y)=0$ are such that $W(\phi_1, \phi_2)(t)\neq 0$ for all t on I and yet ϕ_1, ϕ_2 are linearly dependent on I. Then by the definition of linear dependence there exist constants b_1, b_2 not both zero such that

$$b_1\phi_1(t)+b_2\phi_2(t)=0 \quad \text{for all } t \text{ on } I \tag{3.10}$$

and also

$$b_1\phi_1'(t)+b_2\phi_2'(t)=0 \quad \text{for all } t \text{ on } I \text{ (why?)} \tag{3.11}$$

For each fixed t on I, Eqs. (3.10) and (3.11) are linear homogeneous algebraic equations satisfied by b_1 and b_2, and the determinant of their coefficients is precisely $W(\phi_1, \phi_2)(t)$. Since, by assumption, $W(\phi_1, \phi_2)(t)\neq 0$ at any t on I, it follows from the theory of linear homogeneous systems of algebraic equations (see Appendix 1) that $b_1=b_2=0$, which contradicts the assumed linear dependence of the solutions ϕ_1, ϕ_2 on I. This shows that if the Wronskian of two solutions of $L(y)=0$ is different from zero on I, then these solutions are linearly independent on I.

To prove the second part of the theorem, assume that the solutions ϕ_1, ϕ_2 of $L(y)=0$ are linearly independent on I and assume that there is at least one $\hat{t}$ on I such that $W(\phi_1, \phi_2)(\hat{t})=0$. (If there is no such $\hat{t}$ there is nothing to prove!) Now look again at the algebraic system (3.10), (3.11) for $t=\hat{t}$. It follows, again from the theory of linear homogeneous systems of algebraic equations (see Appendix 1) that, because $W(\phi_1, \phi_2)(\hat{t})=0$, the system of algebraic equations

$$b_1\phi_1(\hat{t})+b_2\phi_2(\hat{t})=0, \quad b_1\phi_1'(\hat{t})+b_2\phi_2'(\hat{t})=0 \tag{3.12}$$

has at least one solution b_1, b_2, where b_1 and b_2 are not both zero. To complete the proof define the function $\psi(t)=b_1\phi_1+b_2\phi_2(t)$, where b_1, b_2 are taken as any solution of (3.12). First observe that ψ is a solution of $L(y)=0$ (why?). Because of (3.12) the solution ψ satisfies the initial conditions $\psi(\hat{t})=0$, $\psi'(\hat{t})=0$. Therefore, by Theorem 1 and Example 2, Section 3.1, $\psi(t)=0$ for every t on I. This means that we have found constants b_1, b_2 not both zero such that $b_1\phi_1(t)+b_2\phi_1(t)=0$ for every t on I. This contradicts the assumed linear independence of the solutions ϕ_1, ϕ_2 on I. Therefore the assumption $W(\phi_1, \phi_2)(\hat{t})=0$ is false; that is, no such $\hat{t}$ exists and $W(\phi_1, \phi_2)(t)\neq 0$ for every t in I. This completes the proof of Theorem 3. ∎

Exercises

12. Show that e^{2t}, e^{-2t} are linearly independent solutions of $y''-4y=0$ on $-\infty<t<\infty$.
13. Show that $e^{-t/2}\cos(\sqrt{3}/2)t$, $e^{-t/2}\sin(\sqrt{3}/2)t$ are linearly independent solutions of $y''+y'+y=0$ on $-\infty<t<\infty$.
14. Show that e^{-t}, te^{-t} are linearly independent solutions of $y''+2y'+y=0$ on $-\infty<t<\infty$.
15. Show that $\sin t^2$, $\cos t^2$ are linearly independent solutions of $ty''-y'+4t^3y=0$ on $0<t<\infty$ or $-\infty<t<0$. Show that $W(\sin t^2, \cos t^2)(0)=0$. Why does this fact not contradict Theorem 3?
16. State the analog of Theorem 3 for the nth-order equation $L_n(y)=a_0(t)\,y^{(n)}+a_1(t)\,y^{(n-1)}+\cdots+a_{n-1}(t)\,y^1+a_n(t)\,y=0$.

We can now establish a result which says that for any two solutions ϕ_1 and ϕ_2 of a linear homogeneous second-order equation with continuous coefficients, the Wronskian is either identically zero or never equal to zero.

Theorem 4. *Let the hypothesis of Theorem 3 be satisfied on some interval I. Let ϕ_1, ϕ_2 be two solutions of $L(y)=0$ on I. Then either their Wronskian $W(\phi_1, \phi_2)(t)$ is zero for every t in I or it is different from zero for every t in I.*

The proof of Theorem 4 is outlined in the following three exercises.

Exercises

17. Let ϕ_1, ϕ_2 be two solutions on some interval I of $L(y)=a_0(t)y''+a_1(t)y'+a_2(t)y=0$, where a_0, a_1, a_2 are continuous on I and $a_0(t)\neq 0$ on I. Show that the Wronskian $W(\phi_1, \phi_2)(t)$ satisfies the first-order linear differential equation

$$W' = -\frac{a_1(t)}{a_0(t)}W, \quad (t \text{ in } I). \qquad (*)$$

[*Hint*: $W'(\phi_1, \phi_2)(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix}' = (\phi_1\phi_2' - \phi_1'\phi_2)' = \phi_1\phi_2'' - \phi_1''\phi_2$.

Now use the fact that ϕ_1, ϕ_2 are solutions of $L(y)=0$ on I to replace ϕ_1'', ϕ_2'' by terms involving $\phi_1, \phi_1', \phi_2, \phi_2'$. Collect terms to obtain (*).]

18. By solving (*) in Exercise 17, derive *Abel's formula*

$$W(\phi_1, \phi_2)(t) = W(\phi_1, \phi_2)(t_0)\exp\left(-\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}ds\right).$$

19. Use the result of Exercise 18 to prove Theorem 4.

20. State and prove the analog of Theorem 4 for the linear third-order differential equation

$$L_3(y) = a_0(t)y''' + a_1(y)y'' + a_2(t)y' + a_3(t)y = 0.$$

21. Show that e^t, $\cos t$, $\sin t$ are linearly independent solutions of the differential equation

$$y''' - y'' + y' - y = 0 \quad \text{on} \quad -\infty < t < \infty.$$

22. Theorem 4, combined with Theorem 3, provides a convenient method for testing solutions of linear differential equations for linear independence on some interval. For, according to these results, it is enough to evaluate the Wronskian at some conveniently chosen point. Thus, for example, show that

$$\phi_1(t) = 1 + \sum_{m=1}^{\infty}\frac{t^{3m}}{2\cdot 3\cdot 5\cdot 6\cdots(3m-1)(3m)}$$

$$\phi_2(t) = t + \sum_{m=1}^{\infty}\frac{t^{3m+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3m)(3m+1)}$$

are linearly independent solutions of $y''-ty=0$ on the interval $-\infty<t<\infty$. (Here you may assume that it has already been shown that ϕ_1 and ϕ_2 are solutions of $y''-ty=0$, but how could you verify this?)

Recall that the linearity of L implies that any linear combination of solutions of $L(y)=0$, is again a solution. We have raised the question: "Can every solution of $L(y)=0$ be generated as a linear combination of some special solution?" We answered this partially in Theorem 1 using a particular pair of linearly independent solutions. With the help of Theorem 3 we can now answer the question completely.

Theorem 5. *Let a_0, a_1, a_2 be functions continuous on some interval I, and let $a_0(t)\neq 0$ for all t in I. If ϕ_1 and ϕ_2 are any two linearly independent solutions of*

$$L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=0$$

on I (not necessarily the two special solutions ϕ_1, ϕ_2 of Theorem 1), then every solution ϕ of $L(y)=0$ on I can be written in the form

$$\phi(t)=c_1\phi_1(t)+c_2\phi_2(t) \qquad t \text{ in } I \tag{3.13}$$

for some unique choice of constants c_1, c_2.

From a practical point of view, the theorem tells us that knowledge (possibly by guessing) of any two linearly independent solutions ϕ_1, ϕ_2 of $L(y)=0$ on I enables us to express every solution by means of Eq. (3.13) by choosing the constants c_1, c_2 suitably. For this reason, we call the function defined by (3.13) *the general solution* of $L(y)=0$ on I, and we sometimes say that the linearly independent solutions form a *fundamental set*.

Proof of Theorem 5. Let ϕ be any a solution of $L(y)=0$ on I and let t_0 be any point in I. Compute $\phi(t_0)=\alpha$, $\phi'(t_0)=\beta$. Because ϕ_1 and ϕ_2 are linearly independent solutions of $L(y)=0$, Theorem 4 tells us that $W(\phi_1, \phi_2)(t)\neq 0$ for all t on I; in particular $W(\phi_1, \phi_2)(t_0)\neq 0$. If the representation (3.13) holds for all t in I, it will have to hold at $t=t_0$. To see if this is possible, we impose the conditions $\phi(t_0)=\alpha$, $\phi'(t_0)=\beta$ and obtain the system of algebraic equations

$$c_1\phi_1(t_0)+c_2\phi_2(t_0)=\alpha, \qquad c_1\phi_1'(t_0)+c_2\phi_2'(t_0)=\beta$$

with determinant of coefficients $W(\phi_1, \phi_2)(t_0)\neq 0$. Therefore by the theory of linear nonhomogeneous systems of algebraic equations (see Appendix 1) this algebraic system can be solved uniquely for c_1, c_2, and we obtain

$$c_1=\frac{\alpha\phi_2'(t_0)-\beta\phi_2(t_0)}{W(\phi_1, \phi_2)(t_0)}, \qquad c_2=\frac{\beta\phi_1(t_0)-\alpha\phi_1'(t_0)}{W(\phi_1, \phi_2)(t_0)}. \tag{3.14}$$

This choice of c_1, c_2 makes (3.13) hold at $t=t_0$. To see whether this choice of

c_1, c_2 does the job for all t in I, we define the function

$$\psi(t)=c_1\phi_1(t)+c_2\phi_2(t),$$

where c_1, c_2 are the numbers given by (3.14). We observe that $\psi(t)$ (as well as $\phi(t)$, $\phi_1(t)$, $\phi_2(t)$) is a solution of $L(y)=0$ on I. To complete the proof we need only show that $\psi(t)=\phi(t)$ for every t in I. But using (3.14) we see that

$$\psi(t_0)=\alpha=\phi(t_0) \quad \text{and} \quad \psi'(t_0)=\beta=\phi'(t_0).$$

Therefore ϕ and ψ are both solutions of $L(y)=0$ on I and they satisfy the same initial conditions at $t=t_0$. By uniqueness (Theorem 1, Section 3.1), ϕ and ψ are identical and this establishes Theorem 5. ∎

Theorem 5 extends easily to higher-order linear differential equations as follows:

Theorem 6. *Let $a_0, a_1, \ldots, a_n$ be functions continuous on some interval I, and let $a_0(t)\neq 0$ on I. Let $\phi_1, \phi_2, \ldots, \phi_n$ be any set of n linearly independent solutions of the equation*

$$L_n(y)=a_0(t)\,y^{(n)}+a_1(t)\,y^{(n-1)}+\cdots+a_{n-1}(t)\,y'+a_n(t)\,y=0,$$

on I. Then every solution ϕ of $L_n(y)=0$ on I can be written as a unique linear combination

$$\phi(t)=c_1\phi_1(t)+\cdots+c_n\phi_n(t)$$

for t on I of the given solutions $\phi_1, \phi_2, \ldots, \phi_n$ (i.e., *there exist unique constants $c_1, c_2, \ldots, c_n$ such that*

$$\phi(t)=c_1\phi_1(t)+c_2\phi_2(t)+\cdots+c_n\phi_n(t).$$

Exercise

23. Prove Theorem 6 if $n=3$.

Remark (for students acquainted with linear algebra). The theory developed in Sections 3.2 and 3.3 shows that the solutions of a linear homogeneous differential equation $L(y)=0$ with continuous coefficients on some interval I and with nonvanishing leading coefficient on I, form a vector space V over the real or complex numbers (see property (i), Section 3.2). Theorem 1 shows that the dimension of V is 2 if L is a linear differential operator of order 2, by exhibiting a basis for V consisting of the special linearly independent solutions ϕ_1 and ϕ_2 constructed in the theorem. Theorem 5 shows that any two linearly independent solutions of $L(y)=0$ also form a basis for V, provided the order of L is 2. We can derive this more simply using knowledge of linear algebra. Once we know, by Theorem 1, that V has dimension 2, it follows immediately that any two linearly independent vectors in V (that is, solutions) span V. Theorem 1 for a homogeneous linear differential equation of order n shows

$$L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=b(t),$$

where a_0, a_1, a_2, and b are real functions defined on some interval I, holds for complex-valued solutions of this equation. This remains true even if a_0, a_1, a_2, and b are complex functions. This specifically applies to existence and uniqueness of such solutions, and linear dependence and independence of such solutions (including the Wronskian test). There is no change needed in any of the statements and their proofs; it is only necessary to bear in mind that the functions which enter each discussion may be complex-valued.

Exercises

4. Show that the functions

$$\exp\left(\frac{-1+\sqrt{3}\,i}{2}\,t\right) \quad \text{and} \quad \exp\left(\frac{-1-\sqrt{3}\,i}{2}\,t\right)$$

satisfy the differential equation $y''+y'+y=0$ for all real t.

5. Reprove Theorem 1, Section 3.3, in the case that the coefficients $a_0(t)$, $a_1(t)$, $a_2(t)$ are continuous complex-valued functions on an interval I and t is real.

We now present a result on complex-valued solutions of *real* linear differential equations which is of great importance in applications. Note that this result is not restricted to equations with constant coefficients.

Suppose that f is any complex-valued function defined in a real interval $\mathscr{I}$. Let $\mathscr{R}f$ denote the real part of f and let $\mathscr{I}f$ denote the imaginary part; e.g., if $f(t)=\exp 2it$ we have

$$(\mathscr{R}f)(t)=\cos 2t \qquad \text{and} \qquad (\mathscr{I}f)(t)=\sin 2t\,.$$

Theorem 1. *Let ϕ be a complex-valued solution of the differential equation*

$$L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=0$$

on some interval I, where a_0, a_1, a_2 are given real functions on I. Then the real functions $u=\mathscr{R}\phi$, $v=\mathscr{I}\phi$ are themselves (real) solutions of $L(y)=0$ on I.

Proof. Since ϕ is a solution of $L(y)=0$ on I, we have

$$a_0(t)\,\phi''(t)+a_1(t)\,\phi'(t)+a_2(t)\,\phi(t)=0$$

for every t on I. (The fact that ϕ may be complex valued does not change anything.) Since $\phi=u+iv$, we have, from the definition of derivative $\phi'(t)=u'(t)+iv'(t)$, $\phi''(t)=u''(t)+iv''(t)$. Therefore

$$a_0(t)\,[u''(t)+iv''(t)]+a_1(t)\,[u'(t)+iv'(t)]+a_2(t)\,[u(t)+iv(t)]=0.$$

Separating the left-hand side into real and imaginary parts, we obtain (remember that a_0, a_1 a_2 are real) for all t on I:

$$a_0(t)\,u''(t)+a_1(t)\,u'(t)+a_2(t)\,u(t)+i[a_0(t)\,v''(t)+a_1(t)\,v'(t)+a_2(t)\,v(t)]=0$$

[*Note*: This also shows that $L(\phi)=L(u)+iL(v)$; this is true in general if L is a linear differential operator with real coefficients.] Since the last relation holds for every t on I and since a complex number is zero if and only if both its real and imaginary parts are zero, we have, for all t in I:

$$L(u)=a_0(t)\,u''(t)+a_1(t)\,u'(t)+a_2(t)\,u(t)=0$$

and

$$L(v)=a_0(t)\,v''(t)+a_1(t)\,v'(t)+a_2(t)\,v(t)=0$$

which shows that $u=\mathscr{R}\phi$ and $v=\mathscr{I}\phi$ are both solutions of $L(y)=0$ on I and completes the proof. ∎

Exercise

6. Let ϕ be a solution on some interval I of the differential equation

$$L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=b(t)$$

where a_0, a_1, a_2 are real and b is complex. Show that $u=\mathscr{R}\phi$ satisfies the equation $L(y)=\mathscr{R}b$ and prove an analogous result for $v=\mathscr{I}\phi$.

Example 2. Use the solutions

$$\phi_1(t)=\exp\left(\frac{-1+\sqrt{3}\,i}{2}\,t\right),\qquad \phi_2(t)=\exp\left(\frac{-1-\sqrt{3}\,i}{2}\,t\right)$$

of the differential equation $y''+y'+y=0$ and Theorem 1 to find the general solution in real form on $-\infty<t<\infty$. They are linearly independent on $-\infty<t<\infty$, since, by Theorem 3, Section 3.3, interpreted for complex-valued solutions,

$$W(\phi_1,\phi_2)(t)=\begin{vmatrix} \exp\left(\frac{-1+\sqrt{3}\,i}{2}\,t\right) & \exp\left(\frac{-1-\sqrt{3}\,i}{2}\,t\right) \\ \frac{-1+\sqrt{3}\,i}{2}\exp\left(\frac{-1+\sqrt{3}\,i}{2}\,t\right) & \frac{-1-\sqrt{3}\,i}{2}\exp\left(\frac{-1-\sqrt{3}\,i}{2}\,t\right) \end{vmatrix}$$

$$=-\sqrt{3}\,ie^{-t}\neq 0,\qquad -\infty<t<\infty.$$

Therefore, by Theorem 5, Section 3.3, interpreted for complex-valued solutions, every solution ϕ (possibly complex valued) of $y''+y'+y=0$ on $-\infty<t<\infty$ has the form $\phi(t)=c_1\phi_1(t)+c_2\phi_2(t)$ for some unique choice of the (possibly complex) constants c_1, c_2. By Theorem 1 (applicable because the coefficients are real) the real functions

$$u_1(t)=\mathscr{R}\phi_1(t)=\exp(-t/2)\cos(\sqrt{3}/2)\,t$$

and

$$v_1(t)=\mathscr{I}\phi_1(t)=\exp(-t/2)\sin(\sqrt{3}/2)\,t$$

are also solutions of $y''+y'+y=0$ for $-\infty<t<\infty$. The same statement applies to

$$u_2(t)=\mathscr{R}\phi_2(t)=\exp(-t/2)\cos(\sqrt{3}/2)\,t$$

and

$$v_2(t)=\mathscr{I}\phi_2(t)=-\exp(-t/2)\sin(\sqrt{3}/2)\,t.$$

You can easily check that $W(u_1, v_1)(t)\neq 0$ on $-\infty<t<\infty$. Therefore, by Theorem 5, Section 3.3, again, every solution ϕ of $y''+y'+y=0$ on $-\infty<t<\infty$ has the form

$$\phi(t)=a_1\exp\left(-\frac{t}{2}\right)\cos\frac{\sqrt{3}}{2}t+a_2\exp\left(-\frac{t}{2}\right)\sin\frac{\sqrt{3}}{2}t$$

for some unique choice of the (possibly complex) constants a_1, a_2. Starting with the complex form of the solution ϕ, we may also arrive at the "real form" as follows. Using Euler's Formula (see Appendix 3) and collecting terms, we have

$$\begin{aligned}\phi(t)&=c_1\phi_1(t)+c_2\phi_2(t)\\&=c_1\exp\left(-\frac{t}{2}\right)\left(\cos\frac{\sqrt{3}}{2}t+i\sin\frac{\sqrt{3}}{2}t\right)\\&\quad+c_2\exp\left(-\frac{t}{2}\right)\left(\cos\frac{\sqrt{3}}{2}t-i\sin\frac{\sqrt{3}}{2}t\right)\\&=(c_1+c_2)\exp\left(-\frac{t}{2}\right)\cos\frac{\sqrt{3}}{2}t+i(c_1-c_2)\exp\left(-\frac{t}{2}\right)\sin\frac{\sqrt{3}}{2}t.\end{aligned}$$

If we now define $a_1=c_1+c_2$, $a_2=i(c_1-c_2)$, we obtain the desired form. It is clear from this that the solution $\phi(t)$ of the equation $y''+y'+y=0$ will be real if and only if $c_2=\overline{c_1}$ (the complex conjugate of c_1). In this case, of course, a_1 and a_2 are both real.

We now return to the general equation $L(y)=0$, where p and q are are real constants, and summarize what we have learned up to this point.

Theorem 2. *Every solution ϕ of the differential equation*

$$y''+py'+qy=0 \tag{3.15}$$

where p, q are real constants with $p^2\neq 4q$ is defined on $-\infty<t<\infty$ and has the form

$$\phi(t)=c_1e^{z_1t}+c_2e^{z_2t},\qquad -\infty<t<\infty \tag{3.17}$$

The numbers z_1, z_2 are the distinct roots of the characteristic equation

$$z^2+pz+q=0 \tag{3.16}$$

and c_1, c_2 are constants. If $p^2>4q$, z_1 and z_2 are real and distinct. If $p^2<4q$ the roots z_1, z_2 are complex conjugates. In this case if $z_1=\alpha+i\beta$ (α, β real) the solution ϕ may be expressed in the form

$$\phi(t)=e^{\alpha t}(a_1\cos\beta t+a_2\sin\beta t) \tag{3.18}$$

where a_1, a_2 are constants. If ϕ is real, a_1 and a_2 are real.