Proof. We have already proved all of Theorem 2 except for Eq. (3.18). To prove (3.18), we proceed exactly as in Exercise 5 above; namely, we know from Theorem 1 that $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$ are solutions of $y''+py'+$ $+qy=0$, where $\alpha+i\beta$ is a root of $z^2+pz+q=0$. Since these solutions are linearly independent on $-\infty<t<\infty$, Eq. (3.18) is a direct consequence of Theorem 5, Section 3.3. ∎

Exercises

7. Show that $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$ are linearly independent solutions on $-\infty<t<\infty$ of (3.15) when $p^2<4q$.
8. Proceeding as in Example 2, show that a_1, a_2 in (3.18) are given in terms of c_1 and c_2 by the formulas $a_1=c_1+c_2$, $a_2=i(c_1-c_2)$, where c_1, c_2 are the constants in (3.17).
9. Find the solution ϕ satisfying the initial conditions $\phi(0)=\phi'(0)=1$ of each of the following differential equations:

   a) $y''+y=0$ b) $y''-4y'+13y=0$
   c) $y''+4y=0$ d) $y''+2y'+2y=0$

*10. In Eq. (3.15) with p, q nonnegative, find conditions on the constants which lead to complex roots of the characteristic equation and investigate the behavior of the solutions for various choices of these constants as $t\to+\infty$.

Theorem 2 enables us to solve Eq. (3.15) completely when the characteristic equation (3.16) has distinct roots.

We now turn to the case of equal roots of the characteristic equation. This occurs if $p^2=4q$, and the characteristic equation

$$z^2+pz+q=0$$

then has the double root $z=-p/2$; therefore $\exp[(-p/2)\,t]$ is a solution of

$$y''+py'+qy=0 \tag{3.15}$$

on $-\infty<t<\infty$ if $p^2=4q$. The theory tells us that in all cases, (3.15) should have two linearly independent solutions. *We now employ a useful trick* to find (guess) a second linearly independent solution. Knowing that $\exp[(-p/2)\,t]$ is a solution of (3.15), we try to determine a nonconstant function w such that

$$\psi(t)=\exp\left(-\frac{p}{2}t\right)w(t) \tag{3.19}$$

will also be a solution of (3.15). Now ψ will be a solution of (3.15) on $-\infty<t<\infty$ if and only if

$$\psi''(t)+p\psi'(t)+q\psi(t)=0$$

or equivalently, using (3.19), if and only if

$$\exp\left(-\frac{p}{2}t\right)w''(t)-p\exp\left(-\frac{p}{2}t\right)w'(t)+\frac{p^2}{4}\exp\left(-\frac{p}{2}t\right)w(t)$$
$$+p\left[-\frac{p}{2}\exp\left(-\frac{p}{2}t\right)w(t)+\exp\left(-\frac{p}{2}t\right)w'(t)\right]+q\exp\left(-\frac{p}{2}t\right)w(t)=0$$

or if and only if

$$\exp\left(-\frac{p}{2}t\right)\left[w''(t)+\left(q-\frac{p^2}{4}\right)w(t)\right]=0, \qquad -\infty<t<\infty.$$

Since $q=p^2/4$, w must satisfy the equation $\exp[(-p/2)t]\,w''(t)=0$. But $\exp[(-p/2)t]\neq 0$, and so (3.19) will be a solution of (3.15) only if w is determined such that $w''(t)=0$ $(-\infty<t<\infty)$. Thus $w(t)=c_1+c_2t$ where c_1, c_2 are constants. We therefore have $\psi(t)=(c_1+c_2t)\exp[(-p/2)t]$ as a candidate for the solution of (3.15). *Direct substitution shows that it is.* Now $\exp[(-p/2)t]$ is a solution of $y''+py'+qy=0$ if $p^2=4q$, and $t\exp[(-p/2)t]$ is another solution on $-\infty<t<\infty$ (verify!). Since they are linearly independent solutions on $-\infty<t<\infty$ (verify!), Theorem 5, Section 3.3, tells us that we have proved the following result.

***Theorem* 3.** *Let p and q be constants such that $p^2=4q$. Then every solution ϕ on $-\infty<t<\infty$ of*

$$y''+py'+qy=0 \tag{3.15}$$

has the form

$$\phi(t)=(c_1+c_2t)\exp\left(-\frac{p}{2}t\right), \qquad -\infty<t<\infty$$

where c_1 and c_2 are constants.

There is an alternative and instructive way to establish the fact that if $p^2=4q$, then $t\exp[(-p/2)t]$ is also a solution of (3.15). We give this method also because it is useful for solving higher-order equations. We know that $\exp[(-p/2)t]$ is a solution of $L(y)=y''+py'+(p^2/4)y=0$. This means that

$$L(e^{zt})\Big|_{z=-p/2}=e^{zt}\left(z^2+pz+\frac{p^2}{2}\right)\Big|_{z=-p/2}=0.$$

Since

$$\frac{\partial}{\partial z}(L(e^{zt}))=\frac{\partial}{\partial z}\left[e^{zt}\left(z^2+pz+\frac{p^2}{4}\right)\right]=te^{zt}\left(z^2+pz+\frac{p^2}{4}\right)+e^{zt}(2z+p)$$

we see by substituting $z=-p/2$ that also

$$\frac{\partial}{\partial z}(L(e^{zt}))\Big|_{z=-p/2}=0.$$

Note that $2z+p$ is the derivative of $z^2+pz+p^2/2$ and both these vanish at the double root $z=-p/2$. (This is a general result about multiple roots—see Appendix 2.) As you may verify,

$$\frac{\partial}{\partial z}\left(\frac{\partial e^{zt}}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial e^{zt}}{\partial z}\right), \qquad \frac{\partial}{\partial z}\left(\frac{\partial^2 e^{zt}}{\partial t^2}\right)=\frac{\partial^2}{\partial t^2}\left(\frac{\partial e^{zt}}{\partial z}\right),$$

so that $(\partial/\partial z)(L(e^{zt}))=L(\partial/\partial z)(e^{zt})$. Therefore

$$L\left(\frac{\partial}{\partial z}e^{zt}\right)\bigg|_{z=-p/2}=L(te^{zt})\bigg|_{z=-p/2}=0.$$

This shows that $te^{(-p/2)(t)}$ is also a solution of $y''+py'+qy=0$ if $q=p^2/4$.

Exercises

11. Find the general solution of each of the following equations. If the equation is real, express the solution in real form. Note that Theorem 3 is true if p and q are complex, and thus equations with complex coefficients can be solved.

    a) $y''+9y=0$
    b) $y''-5y'+6y=0$
    c) $y''+10y'+25y=0$
    d) $y''+2iy'+y=0$
    e) $4y''-y=0$
    f) $y''+5y'+10y=0$
    g) $\varepsilon y''+2y'+y=0, \quad 0<\varepsilon<1$
    h) $4y''+4y'+y=0$

•12. In Eq. (3.15) with $p^2=4q$, investigate the behavior of the solutions as $t\to+\infty$ for various values of the constants.

•13. Recall the *Definition*: *A function f is said to be bounded on some interval I if and only if there exists a constant $M>0$ such that $|f(t)|\le M$ for all t on I.* For example, $\sin t$, $\cos t$ are bounded on any interval, $1/t$ is bounded on $[1, 2]$ but not on $(0, \infty)$, e^{-t} is bounded on $[-5, \infty)$ but not on $(-\infty, -5]$.

    a) Determine which differential equations in Exercise 1 have *all* their solutions bounded on $[0, \infty)$.
    b) Repeat part (a) for the interval $(-\infty, \infty)$.

•14. Show that the solutions of the differential equation $y''+py'+qy=0$, where p and q are positive constants, are oscillations with amplitudes which decrease exponentially when $p^2<4q$ (light damping) and that they decrease exponentially without oscillating if $p^2>4q$ (overdamping). How do they behave if $p^2=4q$ (critical damping)?

The Phase Plane

Let ϕ be a real solution on $0\le t<\infty$ of the linear second-order differential equation $L(y)=0$, where L has real constant coefficients. Let $y_1=\phi(t)$, $y_2=\phi'(t)$, where we now think of t as a parameter, ranging over the

16. a) Write the general solution of the equation $y'' + 2y = 0$ in the "amplitude-phase shift form."

    b) Determine the amplitude, period, phase shift of that solution ϕ of $y'' + 9y = 0$ which satisfies $\phi(0) = 1$, $\phi'(0) = 2$.

    c) Sketch and identify several typical positive semiorbits (that is, let $0 \leq t < \infty$) of the equation $y'' + 9y = 0$ in the phase plane. What happens if we let t range on the interval $-\infty < t \leq 0$ (negative semi-orbit)? Indicate the direction of the motion along each curve as t increases.

17. Sketch a few typical positive semiorbits in the phase plane for each of the following differential equations. Consider also the negative semiorbits. Indicate the direction of the motion along each curve as t increases.

    a) $y'' + 2y' + 2y = 0$

    b) $y'' - y = 0$

18. Suppose we had a pendulum for which a crude mathematical model would give rise either to the equation in Exercise 16a) or 17a) above. Can you give a physical interpretation of the semiorbits in the phase plane in each case?

19. Consider two solutions $\phi(t) = c_1 \cos t + c_2 \sin t$ and $\psi(t) = d_1 \cos t + d_2 \sin t$ of the equation $y'' + y = 0$, where $c_1^2 + c_2^2 = d_1^2 + d_2^2$. Show that these solutions both give rise to the same positive semiorbit in the phase plane, even though the solution ϕ need not be the same as the solution ψ.

Exercise 19 shows that, although only one orbit passes through each point of the phase plane, each orbit corresponds to many solutions with different phase shifts.

3.5 LINEAR HOMOGENEOUS EQUATIONS OF ARBITRARY ORDER WITH CONSTANT COEFFICIENTS

We can easily generalize the results of Section 3.4 for second-order linear differential equations with constant coefficients to equations of arbitrary order. Consider the linear homogeneous equation of order n with constant coefficients $a_1, a_2, \ldots, a_n$

$$L_n(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \tag{3.20}$$

and look for a solution of the form e^{zt} as before. Note that Eq. (3.20) reduces to (3.15) when $n = 2$, with $a_1 = p$, $a_2 = q$. Since $L_n(e^{zt}) = p_n(z)\, e^{zt}$, where

$$p_n(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

is a polynomial of degree n, called the *characteristic polynomial*, we see that the analogs of Theorems 2 and 3, Section 3.4, (although rather more involved) may be stated as follows:

Theorem 1. *Let $z_1, z_2, \ldots, z_s$, where $s \leq n$, be the distinct roots of the characteristic equation* (of degree n)

$$p_n(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = 0$$

and suppose the root z_i *has multiplicity* m_i, $i = 1, \ldots, s$, $(m_1 + m_2 + \cdots + m_s = n)$. *Then the n functions*

$$\begin{array}{l} e^{z_1 t}, te^{z_1 t}, \ldots, t^{m_1 - 1}e^{z_1 t} \\ e^{z_2 t}, te^{z_2 t}, \ldots, t^{m_2 - 1}e^{z_2 t} \\ \cdot \\ \cdot \\ \cdot \\ e^{z_s t}, te^{z_s t}, \ldots, t^{m_s - 1}e^{z_s t} \end{array}$$

are (i) *solutions of* $L_n(y) = 0$ *on* $-\infty < t < \infty$, *and* (ii) *linearly independent on* $-\infty < t < \infty$. *Hence by Theorem 6, Section 3.3, the general solution of* (3.20) *is a linear combination of these n functions.*

Of course, Theorem 1, Section 3.4, holds without change for higher-order equations, and if the coefficients $a_1, \ldots, a_n$ are real each solution of $L_n(y) = 0$ can be expressed in real form exactly as before.

We do not prove Theorem 1 except to remark that if z_i, $1 \le z_i \le s$ is a root of multiplicity m_i of the polynomial equation $p_n(z) = 0$, then

$$p_n(z_i) = 0, \; p_n'(z_i) = 0, \; \ldots, \; p_n^{(m_i - 1)}(z_i) = 0$$

but $p_n^{(m_i)}(z_i) \ne 0$ (see Appendix 3). This observation enables us to prove the result much as was done in Theorem 3, Section 3.4 (alternative approach). The linear independence of these solutions has been proved in Lemma 1, Section 3.3.

Example 1. Find the general solution of the equation $y^{(4)} + 16y = 0$.

Since this equation has order 4, is homogeneous, and has constant coefficients, Theorem 1 is applicable. The characteristic equation is $z^4 + 16 = 0$. To solve this equation, we write

$$z^4 = -16 = 16e^{i(\pi + 2n\pi)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

or, letting $z = re^{i\theta}$,

$$r^4 e^{4i\theta} = 16e^{i(\pi + 2n\pi)}, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Hence $r^4 = 16$ and $\theta = \pi/4 + (n/2)\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, and the distinct roots are $z_1 = 2 \exp[i(\pi/4)] = \sqrt{2}(1+i)$, $z_2 = 2 \exp[3i(\pi/4)] = \sqrt{2}(-1+i)$, $z_3 = 2 \exp[-i(\pi/4)] = \sqrt{2}(1-i)$, $z_4 = 2\exp[-3i(\pi/4)] = \sqrt{2}(-1-i)$, corresponding to $n = 0$, $n = 1$, $n = -1$, $n = -2$, respectively. It is clear that the choices $n = +2, \pm 3, \ldots$, lead us back to one of the roots z_1, z_2, z_3, z_4 already listed. Since $n = 4$ and since the characteristic equation has four distinct roots, every solution ϕ of the equation $y^{(4)} + 16y = 0$ has by Theorem 1 (here $n = 4$, $m_1 = m_2 = m_3 = m_4 = 1$) the form

$$\begin{aligned} \phi(t) = {} & c_1 \exp[\sqrt{2}(1+i)\,t] + c_2 \exp[\sqrt{2}(1-i)\,t] \\ & + c_3 \exp[\sqrt{2}(-1+i)\,t] + c_4 \exp[\sqrt{2}(-1-i)\,t] \end{aligned}$$

for some unique choice of the constants c_1, c_2, c_3, c_4. This may be written in the real form

$$\phi(t)=\exp[\sqrt{2}t](a_1\cos\sqrt{2}t+a_2\sin\sqrt{2}t)+\exp[-\sqrt{2}t](a_3\cos\sqrt{2}t+a_4\sin\sqrt{2}t)$$

for some unique choice of constants a_1, a_2, a_3, a_4.

Example 2. Find the general solution of the equation $y'''+3y''+3y'+y=0$.

Again Theorem 1 is applicable and the characteristic equation is $z^3+3z^2+3z+1=(z+1)^3=0$. Thus $z=-1$ is a triple root and $e^{-t}, te^{-t}, t^2e^{-t}$ are by Theorem 1 linearly independent solutions on $-\infty<t<\infty$. Hence every solution ϕ has the form

$$\phi(t)=e^{-t}(c_1+c_2t+c_3t^2)$$

for some unique choice of the constants c_1, c_2, c_3.

Exercises

1. Find the general solutions of the following differential equations.
   a) $y'''-27y=0$ b) $y^{(4)}-16y=0$
   c) $y^{(4)}+2y''+y=0$ d) $y^{(4)}+5y''+4y=0$
   e) $y^{(6)}+y=0$
2. Find that solution ϕ of $y^{(4)}+16y=0$ for which $\phi(0)=1, \phi'(0)=0, \phi''(0)=0, \phi'''(0)=0$. (See Example 1 above.)
3. Given equation $y^{(4)}+\lambda y=0$, where λ is a constant, find the general solution (in real form) in each case: (a) $\lambda=0$, (b) $\lambda>0$, (c) $\lambda<0$.
4. Which of the equations in Exercise 1 have the property that (a) all their solutions tend to zero as $t\to+\infty$, (b) all their solutions are bounded on $0\le t<\infty$, (c) all their solutions are bounded on $-\infty<t<\infty$?

In closing this section we emphasize that the methods of solution which we have developed are applicable only when the coefficients are constant and the equation is linear! The analog of the phase plane ($n=2$) is n-dimensional phase space and again solutions ϕ of Eq. (3.20) can be pictured as curves in this space. However, we shall not pursue this topic further at this point.

3.6 REDUCTION OF ORDER

nice

The methods of Sections 3.4 and 3.5 do not apply to linear equations with variable coefficients. Thus even though our theory tells us that there are two linearly independent solutions of a second-order linear homogeneous equation, it may not be possible to find them. Sometimes it is possible to guess or by some other means find one solution ϕ_1 of the linear equation

$$L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=0$$

on some interval I where a_0, a_1, a_2 are continuous on I and $a_0(t)\neq 0$ on I. Then the same trick which led to Theorem 3, Section 3.4 (see Eq. 3.19) in the case of constant coefficients with equal roots of the auxiliary equation will enable us to find a second, linearly independent solution of $L(y)=0$ on I by reducing the problem to one of solving a first-order equation.

Assuming that we know a solution ϕ_1, we let $\phi_2(t)=w(t)\,\phi_1(t)$ and try to find a nonconstant function w so that $L(\phi_2)=0$ for every t on I. (Why should w be nonconstant?) Since

$$\phi_2'=w'\phi_1+w\phi_1', \qquad \phi_2''=w''\phi_1+2w'\phi_1'+w\phi_1'',$$

we see that

$$L(\phi_2)=a_0\phi_1 w''+(2a_0\phi_1'+a_1\phi_1)\,w'+wL(\phi_1).$$

But $L(\phi_1)=0$, since ϕ_1 is a solution of $L(y)=0$. Therefore $L(\phi_2)=0$ for all t on I if and only if w satisfies the equation

$$a_0(t)\,\phi_1(t)\,w''+[2a_0(t)\,\phi_1'(t)+a_1(t)\,\phi_1(t)]\,w'=0 \tag{3.21}$$

for all t on I. *Note that* (3.21) *is a first-order linear equation in* w', and is readily solved as follows. Let $w'=v$ and assume that $\phi\neq 0$; then (3.21) becomes

$$v'+\left(2\frac{\phi_1'}{\phi_1}+\frac{a_1}{a_0}\right)v=0.$$

Separating variables and using properties of the logarithm and exponential function, we obtain the solution

$$\begin{aligned} v(t)&=\exp\left[-\int_{t_0}^{t}\left(2\frac{\phi_1'(s)}{\phi_1(s)}+\frac{a_1(s)}{a_0(s)}\right)ds\right]\\ &=\exp\left[-2\log\phi_1(s)\Big|_{s=t_0}^{s=t}-\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}\,ds\right]\\ &=\exp\left(\log[\phi_1(t)]^{-2}-\log[\phi_1(t_0)]^{-2}\right)\exp\left(-\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}\,ds\right)\\ &=\frac{c}{[\phi_1(t)]^2}\exp\left(-\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}\,ds\right), \end{aligned}$$

where c is the constant $[\phi_1(t_0)]^2$. Then, for t_0 and t in I, using $w'=v$, we have

$$w(t)=\int_{t_0}^{t}\frac{c}{[\phi_1(\sigma)]^2}\exp\left(-\int_{t_0}^{\sigma}\frac{a_1(s)}{a_0(s)}\,ds\right)d\sigma,$$

and therefore

$$\phi_2(t) = c\phi_1(t) \int_{t_0}^{t} \frac{1}{[\phi_1(\sigma)]^2} \exp\left(-\int_{t_0}^{\sigma} \frac{a_1(s)}{a_0(s)}\, ds\right) d\sigma. \qquad (3.22)$$

The reader should verify by direct substitution that $L(\phi_2)=0$. This leads us to the following result.

Theorem 1. *If* ϕ_1 *is a solution of* $L(y)=0$ *on* I, *where* a_0, a_1, a_2 *are continuous on* I *and* $a_0(t)\neq 0$ *on* I, *and if* $\phi_1(t)\neq 0$, *then the function* ϕ_2 *given by* (3.21) *is also a solution of* $L(y)=0$ *on* I. *Moreover, the solutions* ϕ_1, ϕ_2 *are linearly independent on* I; *hence every solution* ϕ *of* $L(y)=0$ *on* I *has the form* $\phi = c_1\phi_1 + c_2\phi_2$ *for some unique choice of* c_2, c_2.

We have only to prove the linear independence of the solutions ϕ_1, ϕ_2 on I. This is done by computing $W(\phi_1, \phi_2)$ and using Theorem 3, Section 3.3.

Exercise

1. Carry out the proof of linear independence of ϕ_1, ϕ_2 on I.

Example 1. One solution of $y'' + ty' - y = 0$ is $\phi_1(t) = t$. Find a second linearly independent solution ϕ_2.

We could, of course, apply Eq. (3.21). Rather than try to remember such a complicated formula, we proceed directly by putting $\phi_2 = tw$ and forming $L(\phi_2) = L(tw)$. Since $\phi_2' = tw' + w$ and $\phi_2'' = tw'' + 2w'$ we have $L(\phi_2) = tw'' + 2w' + t^2 w'$. Thus $L(\phi_2)=0$ if and only if $v = w'$ satisfies

$$tv' + (2+t^2)\, v = 0.$$

Separating variables, we obtain

$$v(t) = \exp\left[-\int_{t_0}^{t} \left(\frac{2}{s} + s\right) ds\right].$$

Thus, for $t\neq 0$, $v(t) = c(e^{-t^2/2}/t^2)$, $w(t) = c\int_{t_0}^{t} (e^{-s^2/2}/s^2)\, ds$, $(t_0\neq 0)$, and $\phi_2(t) = ct\int_{t_0}^{t} (e^{-s^2/2}/s^2)\, ds$, $(t_0\neq 0,\ t\neq 0)$, where c is a constant, which we may take to be 1. To establish the linear independence of ϕ_1 and ϕ_2, we form their Wronskian

$$W(\phi_1, \phi_2)(t) = \begin{vmatrix} t & t\displaystyle\int_{t_0}^{t} \frac{e^{-s^2/2}}{s^2}\, ds \\ 1 & \dfrac{e^{-t^2/2}}{t} + \displaystyle\int_{t_0}^{t} \frac{e^{-t^2/2}}{s^2}\, ds \end{vmatrix} = e^{-t^2/2}, \qquad t\neq 0.$$

Since this Wronskian is different from zero, ϕ_1 and ϕ_2 are linearly independent on any interval not containing the origin.

$Q_2 = u(t) - Q_1(t)$

Exercise

2. Given one solution ϕ_1, in each case find a second linearly independent solution ϕ_2 on the interval indicated.

a) $y'' - \frac{2}{t^2} y = 0$ $\phi_1(t) = t^2$, $0 < t < \infty$

b) $y'' - 4ty' + (4t^2 - 2) y = 0$ $\phi_1(t) = e^{t^2}$, $-\infty < t < \infty$

c) $(1 - t^2) y'' - 2ty' + 2y = 0$ $\phi_1(t) = t$, $0 < t < 1$

d) $ty'' - (t+1) y' + y = 0$ $\phi_1(t) = e^t$, $t > 0$.

In general if $\phi_1, \ldots, \phi_k$, where $k < n$, are linearly independent solutions on some interval I of the linear equation of nth order

$$L_n(y) = a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_{n-1}(t)\, y' + a_n(t)\, y = 0.$$

where $a_1, \ldots, a_n$ are continuous on I and $a_0(t) \neq 0$ on I, one can reduce the problem successively to a linear equation of order $n-k$. We illustrate this with the following exercises.

Exercises

3. Suppose ϕ_1, ϕ_2 are linearly independent solutions on an interval I of the differential equation

$$L_3(y) = y''' + a_1(t)\, y'' + a_2(t)\, y' + a_3(t)\, y = 0$$

(we are taking $a_0(t) \equiv 1$).

a) Let $\phi = w\phi_1$ and compute the linear equation of order two which must be satisfied by w' in order that $L_3(\phi) = 0$.

b) Show that $(\phi_2/\phi_1)'$ is a solution of the equation of order two found in part a).

c) Use the result of part b) to reduce the second-order equation to one which is linear and of first order.

4. Two solutions of

$$t^3 y''' - 3ty' + 3y = 0, \qquad t > 0$$

are $\phi_1(t) = t$, $\phi_2(t) = t^3$. Use this and Exercise 3 to find the general solution of the given equation for $t > 0$.

5. a) One solution of the equation

$$L_n(y) = a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_{n-1}(t)\, y' + a_n(t)\, y = 0.$$

is $t^{-1/2} \sin t$. Find the general solution for $t > 0$.

b) Repeat part a) for the equation

$$2ty'' + (1 - 4t)\, y' + (2t - 1)\, y = 0,$$

given that e^t is one solution of the homogeneous equation.

6. Make the change of variably $y=u(t)\,v(t)$ in the equation

$$y''+p(t)\,y'+q(t)\,y=0$$

and choose the function $v(t)$ to make the coefficient of u' in the resulting equation for u equal to zero. Show that the equation for u then becomes

$$u''-\tfrac{1}{4}\{[p(t)]^2+2p'(t)-4q(t)\}\,u=0.$$

7. Apply the change of variable suggested in Exercise 6 to the equation

$$t^2y''+ty'+(t^2-n^2)\,y=0$$

and find the resulting equation.

8. a) Show that the change of variably $y=u'/q(t)\,u$ reduces the nonlinear first-order equation

$$y'+p(t)\,y+q(t)\,y^2=r(t),$$

known as the Riccati equation, to the second-order linear equation

$$u''+\left(p(t)-\frac{q'(t)}{q(t)}\right)u'-r(t)\,q(t)\,u=0.$$

b) Apply this procedure to solve the equation

$$t^2y'+ty+t^2y^2=1.$$

3.7 LINEAR NONHOMOGENEOUS EQUATIONS

We now turn to the nonhomogeneous second-order linear equation

$$L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=f(t) \tag{3.23}$$

and, more generally, the nth-order linear equation

$$L_n(y)=a_0(t)\,y^{(n)}+a_1(t)\,y^{(n-1)}+\cdots+a_n(t)\,y=f(t) \tag{3.24}$$

where throughout $a_0, a_1, \ldots, a_n$, and f are given functions continuous on some interval I, and $a_0(t)\neq 0$ on I. In physical problems having (3.23) or (3.24) for a mathematical model, the nonhomogeneous term $f(t)$ represents an external force acting on the system. For example, if the damped linear mass-spring system considered in Section 2.1 is subjected to a given periodic external force $A\cos\omega t$, then the equation of motion is

$$y''+by'+\frac{k}{m}\,y=\frac{A}{m}\cos\omega t. \tag{3.25}$$

(see Eq. 2.4, Section 2.1). Initial conditions are imposed as before. We remark that the equation for the current in an electrical circuit having resistance, inductance and capacitance in series and a periodic impressed voltage also has the form (3.25) under the appropriate physical assumptions.

The entire development is based on the following fundamental result:

Theorem 1. *Suppose ψ_p is some particular solution of $L(y)=f$ on I, and suppose that ϕ_1, ϕ_2 are two linearly independent solutions of $L(y)=0$ on I. Then every solution ψ of $L(y)=f$ on I has the form*

$$\psi = c_1\phi_1 + c_2\phi_2 + \psi_p \tag{3.26}$$

where c_1, c_2 are constants which can be determined uniquely.

Since every solution of (3.23) has the form (3.26), we refer to (3.26) as *the general solution* of (3.23). According to Theorem 1, to find any solution of (3.23), we need only find two linearly independent solutions ϕ_1, ϕ_2 of $L(y)=0$ and some particular solution ψ_p of $L(y)=f$, and then use the given initial conditions to determine c_1 and c_2.

Proof of Theorem 1. Since ψ_p is a solution of (3.23) on I, we have $L(\psi_p)=f$ for all t on I. Since ψ is also to be a solution of (3.23) on I, we have, using the linearity of L,

$$L(\psi-\psi_p)=L(\psi)-L(\psi_p)=f-f=0.$$

This shows that $\psi-\psi_p$ is a solution of the homogeneous equation $L(y)=0$ on I. (Recall that this much of the proof was already established in Section 3.2 property ii). Therefore, by Theorem 5, Section 3.3, there exist unique constants c_1, c_2 such that

$$\psi-\psi_p=c_1\phi_1+c_2\phi_2 \qquad \text{for all } t \text{ on } I$$

which completes the proof. ∎

The nth-order linear nonhomogeneous equation (3.24) can be treated in the same way, and Theorem 1 has the following analog.

Theorem 2. *Suppose ψ_p is some particular solution of $L_n(y)=f$ on I, and suppose that $\phi_1, \phi_2, \ldots, \phi_n$ are n linearly independent solutions of $L_n(y)=0$ on I. Then every solution ψ of $L_n(y)=f$ on I has the form*

$$\psi=c_1\phi_1+c_2\phi_2+\cdots+c_n\phi_n+\psi_p$$

where $c_1, c_2, \ldots, c_n$ are constants which can be determined uniquely.

Exercises

1. Prove Theorem 2.
2. Compare Theorem 2 in the case $n=1$ with the results of Section 1.4, in particular with Theorem 1, Section 1.4.

We shall now study some methods for finding a particular solution of the equation $L(y)=f$ or $L_n(y)=f$.

The Method of Variation of Constants

This general method for finding a particular solution is applicable whenever one knows the general solution of the associated homogeneous equation $L(y)=0$ or of $L_n(y)=0$ on I. We begin with the second-order case.

Let ϕ_1, ϕ_2 be two linearly independent solutions of $L(y)=0$ on I. (These may either be given to us or in some instances we can find them by one of the methods already studied.) The method consists of finding functions u_1, u_2 such that the function

$$\psi_p = u_1\phi_1 + u_2\phi_2 \tag{3.27}$$

will be forced to satisfy the equation $L(y)=f$ for all t on I. It is remarkable that such a simple device works, because when u_1 and u_2 are constants the function (3.27) satisfies $L(y)=0$ and thus cannot be a solution of $L(y)=f$ for $f\neq 0$. It is of course not obvious that such functions u_1, u_2 can be found. We first argue in reverse; suppose we have found functions u_1, u_2 such that (3.27) satisfies $L(y)=f$ on I. Then we have, for all t on I,

$$(u_1\phi_1+u_2\phi_2)' = u_1\phi_1' + u_2\phi_2' + u_1'\phi_1 + u_2'\phi_2,$$
$$(u_1\phi_1+u_2\phi_2)'' = u_1\phi_1'' + u_2\phi_2'' + 2u_1'\phi_1' + 2u_2'\phi_2' + u_1''\phi_1 + u_2''\phi_2$$

and using $L(\phi_1)=L(\phi_2)=0$ we obtain

$$\begin{aligned} L(u_1\phi_1+u_2\phi_2) &= u_1L(\phi_1)+u_2L(\phi_2)+a_0(\phi_1u_1''+\phi_2u_2'') \\ &\quad +2a_0(\phi_1'u_1'+\phi_2'u_2')+a_1(\phi_1u_1'+\phi_2'u_2') \\ &= a_0[(\phi_1u_1''+\phi_2u_2'')+2(\phi_1'u_1'+\phi_2'u_2')] \\ &\quad +a_1(\phi_1u_1'+\phi_2u_2')=f \end{aligned}$$

for all t on I. We would now like to obtain two relations from which to determine the two functions u_1, u_2. We note that if $\phi_1u_1'+\phi_2u_2'=0$ for all t on I then also $(\phi_1u_1'+\phi_2u_2')'=0$ for all t on I. But $(\phi_1u_1'+\phi_2u_2')'=\phi_1u_1'' + \phi_2u_2''+\phi_1'u_1'+\phi_2'u_2'$. Therefore, if we assume

$$\phi_1u_1'+\phi_2u_2'=0 \tag{3.28}$$

for all t on I, then the requirement

$$a_0[(\phi_1u_1''+\phi_2u_2'')+2(\phi_1'u_1'+\phi_2'u_2')]+a_1(\phi_1u_1'+\phi_2u_2')=f$$

implies, on using (3.28) and the equation obtained by differentiating (3.28), that we must also have, since (3.28) implies that $\phi_1u_1''+\phi_2u_2''+\phi_1'u_1' + \phi_2'u_2'=0$,

$$\phi_1'u_1'+\phi_2'u_2'=\frac{f}{a_0} \tag{3.29}$$

for all t on I. Thus the assumption of the existence of a solution of the form (3.27) of the equation $L(y)=f$ has led us to the two equations (3.28), (3.29) from which we hope to determine u_1', u_2' and then the functions

u_2. But now reversing the argument we see that *if we can find two functions u_1', u_2' to satisfy equations* (3.28), (3.29), then indeed $\psi_p = u_1\phi_1 + u_2\phi_2$ will satisfy $L(y) = f$ on I.

To find a particular solution of the equation $L(y) = f$, we may therefore concentrate on equations (3.28), (3.29). These are linear algebraic equations for the quantities u_1', u_2' and the determinant of their coefficients is $W(\phi_1, \phi_2)$. Since the solutions ϕ_1, ϕ_2 of $L(y) = 0$ are by hypothesis linearly independent on I, it follows that $W(\phi_1, \phi_2)(t) \neq 0$ for all t on I (Theorem 2, Section 3.3) and the system ((3.28), (3.29)) of equations can therefore be always solved (in fact uniquely) for the quantities u_1', u_2'. By Cramer's rule (Appendix 1), the solution of the algebraic equations (3.28), (3.29) is

$$u_1' = \frac{-f\phi_2}{a_0 W(\phi_1, \phi_2)}, \quad u_2' = \frac{f\phi_1}{a_0 W(\phi_1, \phi_2)}, \quad t \text{ on } I.$$

Thus a possible choice for u_1, u_2 is

$$u_1(t) = -\int_{t_0}^{t} \frac{f(s)\,\phi_2(s)}{a_0(s)\,W(\phi_1, \phi_2)(s)}\,ds, \quad u_2(t) = \int_{t_0}^{t} \frac{f(s)\,\phi_1(s)}{a_0(s)\,W(\phi_1, \phi_2)(s)}\,ds$$

for any t_0, t in I, where we have taken the constant of integration to be zero. Substituting in (3.27), we find that

$$\psi_p(t) = \int_{t_0}^{t} \frac{f(s)\,[\phi_2(t)\,\phi_1(s) - \phi_1(t)\,\phi_2(s)]}{a_0(s)\,W(\phi_1, \phi_2)(s)}\,ds \tag{3.30}$$

is a solution of $L(y) = f$ on I, as may be verified by direct substitution. We have thus sketched the derivation of the following important result.

Theorem 3. *Let ϕ_1, ϕ_2 be any two linearly independent solutions of the equation*

$$L(y) = a_0(t)\,y'' + a_1(t)\,y' + a_2(t)\,y = 0,$$

where a_0, a_1, a_2 are continuous functions on some interval I and $a_0(t) \neq 0$ on I. Then a particular solution ψ_p of $L(y) = f$, where f is continuous on I, is given by Eq. (3.30).

Equation (3.30) is usually called the *variation-of-constants formula*. The reason for this name is clear from the method. Although the condition expressed by Eq. (3.28) is artificial, the fact that we can solve the problem using it justifies it, and this is actually the essence of the method. This method of finding a particular solution of $L(y) = f$ can be used whenever the coefficients a_0, a_1, a_2 in L and the function f are continuous on I and $a_0 \neq 0$ on I, and whenever one knows the general solution of the associated homogeneous equation. It is not restricted to equations with constant coefficients.

Exercise

3. Prove Theorem 3 by direct substitution of the function ψ_p given by (3.30) into $L(y)=f$ on I. [*Hint*: Write ψ_p in the form

$$\psi_p(t)=\phi_2(t)\int_{t_0}^{t}\frac{f(s)\,\phi_1(s)}{a_0(s)\,W(\phi_1,\phi_2)(s)}\,ds-\phi_1(t)\int_{t_0}^{t}\frac{f(s)\,\phi_2(s)}{a_0(s)\,W(\phi_1,\phi_2)(s)}\,ds$$

before beginning the differentiation.]

Example 1. Find the general solution of the equation

$$y''+y=\tan t,\qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

Since, by Section 3.4, $\phi_1(t)=\cos t$, $\phi_2(t)=\sin t$ are linearly independent solutions of $y''+y=0$ on any interval, they are linearly independent on $-\pi/2<t<\pi/2$; in fact

$$W(\phi_1,\phi_2)(t)=\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}\equiv 1.$$

Instead of memorizing Theorem 3, it is simpler to remember the key steps of the method. By what we have just seen, $\psi_p=u_1\cos t+u_2\sin t$ will be a solution of $y''+y=\tan t$ on $-\pi/2<t<\pi/2$ if and only if the functions u_1 and u_2 are such that u_1', u_2' satisfy Eqs. (3.28), (3.29), that is, if and only if

$$\begin{cases} u_1'\cos t+u_2'\sin t=0\\ -u_1'\sin t+u_2'\cos t=\tan t\end{cases}\qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

Thus

$$u_1'(t)=-\tan t\,\sin t=-\frac{\sin^2 t}{\cos t}=-\frac{1-\cos^2 t}{\cos t}=\cos t-\sec t$$

$$u_2'(t)=\cos t\,\tan t=\sin t,\qquad -\frac{\pi}{2}<t<\frac{\pi}{2},$$

and we may take

$$u_1(t)=\int_0^t(\cos t-\sec t)\,dt,\qquad u_2(t)=\int_0^t\sin t\,dt$$

or

$$u_1(t)=\sin t-\log|\sec t+\tan t|,\qquad u_2(t)=-\cos t,\qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

Therefore $\psi_p=(\sin t-\log(|\sec t+\tan t|)\cos t+(-\cos t)\sin t=-\cos t\,\log(|\sec t+\tan t|)$.
Therefore, by Theorem 1, every solution of $y''+y=\tan t$ on $-\pi/2<t<\pi/2$ has the form

$$\psi(t)=c_1\cos t+c_2\sin t-\cos t\log|\sec t+\tan t|$$

for some unique choice of the constants c_1 and c_2.

Exercises

4. Find the general solution of each of the following differential equations.

   a) $y''+y=\sec t, \quad -\pi/2<t<\pi/2$

   b) $y''+4y'+4y=\cos 2t$

   c) $y''+4y=f(t)$, where f is any continuous function on some interval I

   d) $y''-4y'+4y=3e^{-t}+2t^2+\sin t$

   e) $y''+(1/4t^2)y=f(t)$, $(t>0)$, f continuous, given that $\phi_1(t)=\sqrt{t}$ is a solution of the homogeneous equation.

5. Given that ϕ is a solution of the equation $y''+k^2y=f(t)$, where k is a real constant different from zero and f is continuous for $0\le t<\infty$, show that c_1 and c_2 can be chosen so that

$$\phi(t)=c_1\cos kt+\frac{c_2}{k}\sin kt+\frac{1}{k}\int_0^t \sin k(t-s)\, f(s)\, ds$$

for $0\le t<\infty$. (Use $\cos kt$ and $\sin kt/k$ as a fundamental set of solutions of the homogeneous equation.) Find an analogous formula in the case $k=0$.

6. Given the equation

$$y''+5y'+4y=f(t),$$

use the variation-of-constants formula and Theorem 1 to prove that:

   a) If f is bounded on $0\le t<\infty$ (that is, there exists a constant $M\ge 0$ such that $|f(t)|\le M$ on $0\le t<\infty$), then every solution of $y''+5y'+4y=f(t)$ is bounded on $0\le t<\infty$.

   b) If also $f(t)\to 0$ as $t\to\infty$, then every solution ϕ of $y''+5y'+4y=f(t)$ satisfies $\phi(t)\to 0$ as $t\to\infty$.

* 7. Can you formulate Exercise 6 for the general equation

$$y''+a_1y'+a_2y=f(t), \qquad a_1a_2 \text{ constant}$$

with a_1, a_2 suitably restricted?

The method of variation of constants and Theorem 2 are applicable to the nth-order equation $L_n(y)=f$ with coefficients $a_0, a_1, \ldots a_n$, f continuous and $a_0(t)\ne 0$ on some interval I *provided one knows n linearly independent solutions $\phi_1, \phi_2, \ldots, \phi_n$ of the homogeneous equation $L_n(y)=0$ on I.* Using the second-order case for motivation, we try to find n functions $u_1, u_2, \ldots, u_n$, not all constant, so that

$$\psi_p=u_1\phi_1+u_2\phi_2+\cdots+u_n\phi_n$$

will be a solution of $L_n(y)=f$ on I. If (see the second-order case) $u_1'\phi_1+u_2'\phi_2+\ldots+u_n'\phi_n=0$ on I, then $\psi_p'=u_1\phi_1'+\cdots+u_n\phi_n'$ on I and if $u_1'\phi_1'+\cdots+u_n'\phi_n'=0$ on I, then $\psi_p''=u_1\phi_1''+\cdots+u_n\phi_n''$. Continuing in this manner we find that if $u_1', u_2', \ldots, u_n'$ are chosen to satisfy the system of linear algebraic equations on I

$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 + \cdots + u_n'\phi_n &= 0 \\ u_1'\phi_1' + u_2'\phi_2' + \cdots + u_n'\phi_n' &= 0 \\ &\vdots \\ u_1'\phi_1^{(n-2)} + u_2'\phi_2^{(n-2)} + \cdots + u_n'\phi_n^{(n-2)} &= 0 \\ u_1'\phi_1^{(n-1)} + u_2'\phi_2^{(n-1)} + \cdots + u_n'\phi_n^{(n-1)} &= \frac{f}{a_0} \end{aligned} \tag{3.31}$$

then the function

$$\psi_p = u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n$$

will satisfy $L_n(y) = f$ on I.

Note that the determinant of coefficients of the system of equations (3.31) is $W(\phi_1, \phi_2, \ldots, \phi_n)$, which is different from zero for every t in I since $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent solutions of $L_n(y) = 0$ on I.

Exercise

• 8. Verify that the function $\psi_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$ is a solution of $L_3(y) = f$ on I. [*Hint*: Solve (3.31) by Cramer's rule and integrate to find u_1, u_2, u_3.]

Thus the entire problem is reduced to solving the algebraic system (3.31). Since its determinant of coefficients is $W(\phi_1, \phi_2, \ldots, \phi_n)$, and since the solutions $\phi_1, \ldots, \phi_n$ of $L_n(y) = 0$ on I are linearly independent on I, $W(\phi_1, \ldots, \phi_n)(t) \neq 0$ for t on I and (3.31) always has a unique solution for the quantities $u_1', \ldots, u_n'$ on I. In fact, letting $W_j(t)$ be the n by n determinant having the same elements as $W(\phi_1, \ldots, \phi_n)(t)$ except with $(0, 0, \ldots, 1)$ as its jth column, we see that Cramer's rule (Appendix 1) gives

$$u_j'(t) = \frac{W_j(t)}{W(\phi_1, \ldots, \phi_n)} \frac{f(t)}{a_0(t)}, \qquad j = 1, \ldots, n.$$

The u_j are obtained by integration, so that

$$\psi_p(t) = \sum_{j=1}^{n} \phi_j(t) \int_{t_0}^{t} \frac{W_j(s)}{W(\phi_1, \ldots, \phi_n)(s)} \frac{f(s)}{a_0(s)}\, ds, \tag{3.32}$$

where t_0 and t are any two points of I.

We have therefore sketched the derivation of the following result, which generalizes Theorem 3.

Theorem 4. *Let $\phi_1, \phi_2, \ldots, \phi_n$ be n linearly independent solutions of the equation*

CHAPTER 4

Linear Systems of Differential Equations

In this chapter we will study the theory of systems of linear differential equations, together with an outline of the basic theory of nonlinear systems at the end of the chapter. As we shall see presently, mathematical models of physical systems somewhat more complicated than those investigated in Chapters 1, 2, 3 (motion of several interacting particles, electrical networks, population problems involving more than one species, etc.) often lead to systems of more than one differential equation. Such systems can be reduced in most instances to *linear systems* of first-order differential equations with the aid of certain simplifying assumptions. To study these linear systems, we shall make use of linear algebra (vector spaces and matrix algebra). As a very special case of every result obtained in this chapter we will obtain a corresponding result for a scalar linear differential equation of second (or higher) order, such as those studied in Chapter 3. *Thus students familiar with linear algebra can study this development directly and omit Chapter 3.*

4.1 INTRODUCTION

We shall consider *systems of first-order linear differential equations* of the form

$$\begin{aligned} y_1' &= a_{11}(t)\, y_1 + a_{12}(t)\, y_2 + \cdots + a_{1n}(t)\, y_n + g_1(t) \\ y_2' &= a_{21}(t)\, y_1 + a_{22}(t)\, y_2 + \cdots + a_{2n}(t)\, y_n + g_2(t) \\ &\;\;\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ y_n' &= a_{n1}(t)\, y_1 + a_{n2}(t)\, y_2 + \cdots + a_{nn}(t)\, y_n + g_n(t), \end{aligned} \tag{4.1}$$

where the given functions $a_{ij}(t)$, where $i, j = 1, \ldots, n$, and $g_i(t)$, where $i = 1, \ldots, n$, are continuous on some fixed interval $\mathscr{I}$. Unless mentioned specifically otherwise the interval $\mathscr{I}$ can be open, closed, half open, finite or infinite. If $n=1$, we have the important special case of a scalar first-order equation, which we write in the form

$$y' = p(t)\, y + q(t), \tag{4.2}$$

where $p(t)$ and $q(t)$ are given functions.

System (4.1) is linear in $y_1, y_2, \ldots, y_n$, and $y'_1, y'_2, \ldots, y'_n$. The scalar equation $y' = 2y^2$ is an example of a nonlinear differential equation. These much more complicated equations will not be considered in this chapter; of course some simple cases of nonlinear equations have been studied in Chapter 1.

Example 1. Consider the system

$$\begin{aligned} y'_1 &= y_1 - t y_2 + e^t \\ y'_2 &= t^2 y_1 - y_3 \\ y'_3 &= y_1 + y_2 - y_3 + 2e^{-t}, \end{aligned} \tag{4.3}$$

where $\mathscr{I}$ is the real line, $\{t \mid -\infty < t < \infty\}$. Here, $n=3$, and in the notation of (6.1)

$$\begin{array}{lll|l} a_{11}(t)=1 & a_{12}(t)=-t & a_{13}(t)=0 & g_1(t)=e^t \\ a_{21}(t)=t^2 & a_{22}(t)=0 & a_{23}(t)=-1 & g_2(t)=0 \\ a_{31}(t)=1 & a_{32}(t)=1 & a_{33}(t)=-1 & g_3(t)=2e^{-t}. \end{array}$$

Consider now the array

$$A(t) = \begin{bmatrix} 1 & -t & 0 \\ t^2 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}. \tag{4.4}$$

$A(t)$ is a matrix whose entries are functions. The properties of matrix addition, multiplication by scalars, and matrix multiplication with constant entries also hold for matrices whose entries are functions defined on a common interval $\mathscr{I}$. Let $\mathbf{y}$ and $\mathbf{y}'$ be the column vectors

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix},$$

and let $\mathbf{g}(t)$ be the vector

$$\mathbf{g}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^{-t} \end{bmatrix}. \tag{4.5}$$

Then, observing that matrix vector multiplication of $A(t)$ and $\mathbf{y}$ gives

$$A(t)\,\mathbf{y} = \begin{bmatrix} y_1 - t y_2 \\ t^2 y_1 - y_3 \\ y_1 + y_2 - y_3 \end{bmatrix}$$

we see that system (4.3) may be represented conveniently in the matrix vector form

$$\mathbf{y}' = A(t)\,\mathbf{y} + \mathbf{g}(t),$$

where $A(t)$ and $\mathbf{g}(t)$ are given respectively by (4.4) and (4.5).

Returning to the general case of system (4.1), we define the $n\times n$ matrix

$$A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \tag{4.6}$$

whose entries are the n^2 functions $a_{ij}(t)$, where $i, j=1, \ldots, n$. Next, define the vectors $\mathbf{g}(t)$, $\mathbf{y}$, $\mathbf{y}'$ by the relations

$$\mathbf{g}(t)=\begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}, \quad \mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{y}'=\begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{bmatrix}. \tag{4.7}$$

Then the system (4.1) can be written in the form

$$\mathbf{y}'=A(t)\,\mathbf{y}+\mathbf{g}(t). \tag{4.8}$$

Exercise

1. Given the system

$$\begin{cases} y'_1=y_2+\cos t \\ y'_2=y_1 \end{cases}$$

Define the matrix $A(t)$ and the vectors $\mathbf{y}$, $\mathbf{y}'$, $\mathbf{g}(t)$, and write this system in the form (4.8).

Before proceeding with the definition of a solution and a discussion of the system (4.8), we need the following definitions.

Definition 1. *A matrix* (such as $A(t)$) *or a vector* (such as $\mathbf{g}(t)$) *is continuous on an interval* $\mathscr{I}$ *if and only if each of its entries is a continuous function at each point of* $\mathscr{I}$.

Definition 2. *An* $n\times n$ *matrix* $B(t)$ *or a vector* $\mathbf{u}(t)$ *with* n *components, defined on an interval* $\mathscr{I}$ *and given respectively by*

$$B(t)=\begin{bmatrix} b_{11}(t) & b_{12}(t) & \cdots & b_{1n}(t) \\ b_{21}(t) & b_{22}(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & & \vdots \\ b_{n1}(t) & b_{n2}(t) & \cdots & b_{nn}(t) \end{bmatrix}, \quad \mathbf{u}(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix},$$

is differentiable on $\mathscr{I}$ *if and only if each of its entries is differentiable at every point of* $\mathscr{I}$. *Their derivatives are given by*

$$B'(t)=\begin{bmatrix} b'_{11}(t) & b'_{12}(t) & \cdots & b'_{1n}(t) \\ b'_{21}(t) & b'_{22}(t) & \cdots & b'_{2n}(t) \\ \vdots & \vdots & & \vdots \\ b'_{n1}(t) & b'_{n2}(t) & \cdots & b'_{nn}(t) \end{bmatrix}, \quad \mathbf{u}'(t)=\begin{bmatrix} u'_1(t) \\ u'_2(t) \\ \vdots \\ u'_n(t) \end{bmatrix},$$

respectively. Similarly, the matrix $B(t)$ or the vector $\mathbf{u}(t)$ is integrable on an interval (c, d) if and only if each of its entries is integrable on the interval (c, d). Their integrals are given by

$$\int_c^d B(t)\,dt = \begin{bmatrix} \int_c^d b_{11}(t)\,dt & \int_c^d b_{12}(t)\,dt & \cdots & \int_c^d b_{1n}(t)\,dt \\ \int_c^d b_{21}(t)\,dt & \int_c^d b_{22}(t)\,dt & \cdots & \int_c^d b_{2n}(t)\,dt \\ \vdots & \vdots & & \vdots \\ \int_c^d b_{n1}(t)\,dt & \int_c^d b_{n2}(t)\,dt & \cdots & \int_c^d b_{nn}(t)\,dt \end{bmatrix}$$

$$\int_c^d \mathbf{u}(t)\,dt = \begin{bmatrix} \int_c^d u_1(t)\,dt \\ \int_c^d u_2(t)\,dt \\ \vdots \\ \int_c^d u_n(t)\,dt \end{bmatrix}$$

Exercises

2. Evaluate the derivatives of each of the following vectors or matrices:

a) $B(t) = \begin{bmatrix} t & e^{-t} & 7 \\ \sin t & 0 & \cos t \\ t^2 & t & 1 \end{bmatrix}$ for $-\infty < t < \infty$.

b) $B(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ for $-\infty < t < \infty$.

c) $B(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)\,e^{2t} \end{bmatrix}$ for $-\infty < t < \infty$.

d) $\mathbf{u}(t) = \begin{bmatrix} \log t \\ t \log t \\ t^2 \log t \end{bmatrix}$ for $0 < t < \infty$ (and where $\log t$ is the natural logarithm of t).