The entire development is based on the following fundamental result:

Theorem 1. *Suppose ψ_p is some particular solution of $L(y)=f$ on I, and suppose that ϕ_1, ϕ_2 are two linearly independent solutions of $L(y)=0$ on I. Then every solution ψ of $L(y)=f$ on I has the form*

$$\psi=c_1\phi_1+c_2\phi_2+\psi_p \tag{3.26}$$

where c_1, c_2 are constants which can be determined uniquely.

Since every solution of (3.23) has the form (3.26), we refer to (3.26) as *the general solution* of (3.23). According to Theorem 1, to find any solution of (3.23), we need only find two linearly independent solutions ϕ_1, ϕ_2 of $L(y)=0$ and some particular solution ψ_p of $L(y)=f$, and then use the given initial conditions to determine c_1 and c_2.

Proof of Theorem 1. Since ψ_p is a solution of (3.23) on I, we have $L(\psi_p)=f$ for all t on I. Since ψ is also to be a solution of (3.23) on I, we have, using the linearity of L,

$$L(\psi-\psi_p)=L(\psi)-L(\psi_p)=f-f=0.$$

This shows that $\psi-\psi_p$ is a solution of the homogeneous equation $L(y)=0$ on I. (Recall that this much of the proof was already established in Section 3.2 property ii). Therefore, by Theorem 5, Section 3.3, there exist unique constants c_1, c_2 such that

$$\psi-\psi_p=c_1\phi_1+c_2\phi_2 \qquad \text{for all } t \text{ on } I$$

which completes the proof. ∎

The nth-order linear nonhomogeneous equation (3.24) can be treated in the same way, and Theorem 1 has the following analog.

Theorem 2. *Suppose ψ_p is some particular solution of $L_n(y)=f$ on I, and suppose that $\phi_1, \phi_2, \ldots, \phi_n$ are n linearly independent solutions of $L_n(y)=0$ on I. Then every solution ψ of $L_n(y)=f$ on I has the form*

$$\psi=c_1\phi_1+c_2\phi_2+\cdots+c_n\phi_n+\psi_p$$

where $c_1, c_2, \ldots, c_n$ are constants which can be determined uniquely.

Exercises

1. Prove Theorem 2.
2. Compare Theorem 2 in the case $n=1$ with the results of Section 1.4, in particular with Theorem 1, Section 1.4.

We shall now study some methods for finding a particular solution of the equation $L(y)=f$ or $L_n(y)=f$.

The Method of Variation of Constants

This general method for finding a particular solution is applicable whenever one knows the general solution of the associated homogeneous equation $L(y)=0$ or of $L_n(y)=0$ on I. We begin with the second-order case.

Let ϕ_1, ϕ_2 be two linearly independent solutions of $L(y)=0$ on I. (These may either be given to us or in some instances we can find them by one of the methods already studied.) The method consists of finding functions u_1, u_2 such that the function

$$\psi_p = u_1\phi_1 + u_2\phi_2 \tag{3.27}$$

will be forced to satisfy the equation $L(y)=f$ for all t on I. It is remarkable that such a simple device works, because when u_1 and u_2 are constants the function (3.27) satisfies $L(y)=0$ and thus cannot be a solution of $L(y)=f$ for $f\neq 0$. It is of course not obvious that such functions u_1, u_2 can be found. We first argue in reverse; suppose we have found functions u_1, u_2 such that (3.27) satisfies $L(y)=f$ on I. Then we have, for all t on I,

$$(u_1\phi_1+u_2\phi_2)' = u_1\phi_1' + u_2\phi_2' + u_1'\phi_1 + u_2'\phi_2,$$
$$(u_1\phi_1+u_2\phi_2)'' = u_1\phi_1'' + u_2\phi_2'' + 2u_1'\phi_1' + 2u_2'\phi_2' + u_1''\phi_1 + u_2''\phi_2$$

and using $L(\phi_1)=L(\phi_2)=0$ we obtain

$$\begin{aligned} L(u_1\phi_1+u_2\phi_2) &= u_1L(\phi_1)+u_2L(\phi_2)+a_0(\phi_1u_1''+\phi_2u_2'') \\ &\quad +2a_0(\phi_1'u_1'+\phi_2'u_2')+a_1(\phi_1u_1'+\phi_2'u_2') \\ &= a_0[(\phi_1u_1''+\phi_2u_2'')+2(\phi_1'u_1'+\phi_2'u_2')] \\ &\quad +a_1(\phi_1u_1'+\phi_2u_2')=f \end{aligned}$$

for all t on I. We would now like to obtain two relations from which to determine the two functions u_1, u_2. We note that if $\phi_1u_1'+\phi_2u_2'=0$ for all t on I then also $(\phi_1u_1'+\phi_2u_2')'=0$ for all t on I. But $(\phi_1u_1'+\phi_2u_2')'=\phi_1u_1'' + \phi_2u_2''+\phi_1'u_1'+\phi_2'u_2'$. Therefore, if we assume

$$\phi_1u_1'+\phi_2u_2'=0 \tag{3.28}$$

for all t on I, then the requirement

$$a_0[(\phi_1u_1''+\phi_2u_2'')+2(\phi_1'u_1'+\phi_2'u_2')]+a_1(\phi_1u_1'+\phi_2u_2')=f$$

implies, on using (3.28) and the equation obtained by differentiating (3.28), that we must also have, since (3.28) implies that $\phi_1u_1''+\phi_2u_2''+\phi_1'u_1' + \phi_2'u_2'=0$,

$$\phi_1'u_1'+\phi_2'u_2'=\frac{f}{a_0} \tag{3.29}$$

for all t on I. Thus the assumption of the existence of a solution of the form (3.27) of the equation $L(y)=f$ has led us to the two equations (3.28), (3.29) from which we hope to determine u_1', u_2' and then the functions

u_2. But now reversing the argument we see that *if we can find two functions u_1', u_2' to satisfy equations* (3.28), (3.29), then indeed $\psi_p = u_1\phi_1 + u_2\phi_2$ will satisfy $L(y)=f$ on I.

To find a particular solution of the equation $L(y)=f$, we may therefore concentrate on equations (3.28), (3.29). These are linear algebraic equations for the quantities u_1', u_2' and the determinant of their coefficients is $W(\phi_1, \phi_2)$. Since the solutions ϕ_1, ϕ_2 of $L(y)=0$ are by hypothesis linearly independent on I, it follows that $W(\phi_1, \phi_2)(t)\neq 0$ for all t on I (Theorem 2, Section 3.3) and the system ((3.28), (3.29)) of equations can therefore be always solved (in fact uniquely) for the quantities u_1', u_2'. By Cramer's rule (Appendix 1), the solution of the algebraic equations (3.28), (3.29) is

$$u_1' = \frac{-f\phi_2}{a_0 W(\phi_1, \phi_2)}, \quad u_2' = \frac{f\phi_1}{a_0 W(\phi_1, \phi_2)}, \quad t \text{ on } I.$$

Thus a possible choice for u_1, u_2 is

$$u_1(t) = -\int_{t_0}^{t} \frac{f(s)\,\phi_2(s)}{a_0(s)\,W(\phi_1, \phi_2)(s)}\,ds, \quad u_2(t) = \int_{t_0}^{t} \frac{f(s)\,\phi_1(s)}{a_0(s)\,W(\phi_1, \phi_2)(s)}\,ds$$

for any t_0, t in I, where we have taken the constant of integration to be zero. Substituting in (3.27), we find that

$$\psi_p(t) = \int_{t_0}^{t} \frac{f(s)\,[\phi_2(t)\,\phi_1(s) - \phi_1(t)\,\phi_2(s)]}{a_0(s)\,W(\phi_1, \phi_2)(s)}\,ds \tag{3.30}$$

is a solution of $L(y)=f$ on I, as may be verified by direct substitution. We have thus sketched the derivation of the following important result.

Theorem 3. *Let ϕ_1, ϕ_2 be any two linearly independent solutions of the equation*

$$L(y) = a_0(t)\,y'' + a_1(t)\,y' + a_2(t)\,y = 0,$$

where a_0, a_1, a_2 are continuous functions on some interval I and $a_0(t)\neq 0$ on I. Then a particular solution ψ_p of $L(y)=f$, where f is continuous on I, is given by Eq. (3.30).

Equation (3.30) is usually called the *variation-of-constants formula*. The reason for this name is clear from the method. Although the condition expressed by Eq. (3.28) is artificial, the fact that we can solve the problem using it justifies it, and this is actually the essence of the method. This method of finding a particular solution of $L(y)=f$ can be used whenever the coefficients a_0, a_1, a_2 in L and the function f are continuous on I and $a_0\neq 0$ on I, and whenever one knows the general solution of the associated homogeneous equation. It is not restricted to equations with constant coefficients.

Exercise

3. Prove Theorem 3 by direct substitution of the function ψ_p given by (3.30) into $L(y)=f$ on I. [*Hint*: Write ψ_p in the form

$$\psi_p(t)=\phi_2(t)\int_{t_0}^{t}\frac{f(s)\,\phi_1(s)}{a_0(s)\,W(\phi_1,\phi_2)(s)}\,ds-\phi_1(t)\int_{t_0}^{t}\frac{f(s)\,\phi_2(s)}{a_0(s)\,W(\phi_1,\phi_2)(s)}\,ds$$

before beginning the differentiation.]

Example 1. Find the general solution of the equation

$$y''+y=\tan t,\qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

Since, by Section 3.4, $\phi_1(t)=\cos t$, $\phi_2(t)=\sin t$ are linearly independent solutions of $y''+y=0$ on any interval, they are linearly independent on $-\pi/2<t<\pi/2$; in fact

$$W(\phi_1,\phi_2)(t)=\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}\equiv 1.$$

Instead of memorizing Theorem 3, it is simpler to remember the key steps of the method. By what we have just seen, $\psi_p=u_1\cos t+u_2\sin t$ will be a solution of $y''+y=\tan t$ on $-\pi/2<t<\pi/2$ if and only if the functions u_1 and u_2 are such that u_1', u_2' satisfy Eqs. (3.28), (3.29), that is, if and only if

$$\begin{cases} u_1'\cos t+u_2'\sin t=0\\ -u_1'\sin t+u_2'\cos t=\tan t\end{cases}\qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

Thus

$$u_1'(t)=-\tan t\sin t=-\frac{\sin^2 t}{\cos t}=-\frac{1-\cos^2 t}{\cos t}=\cos t-\sec t$$

$$u_2'(t)=\cos t\tan t=\sin t,\qquad -\frac{\pi}{2}<t<\frac{\pi}{2},$$

and we may take

$$u_1(t)=\int_0^t(\cos t-\sec t)\,dt,\qquad u_2(t)=\int_0^t\sin t\,dt$$

or

$$u_1(t)=\sin t-\log|\sec t+\tan t|,\qquad u_2(t)=-\cos t,\qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

Therefore $\psi_p=(\sin t-\log(|\sec t+\tan t|)\cos t+(-\cos t)\sin t=-\cos t\,\log(|\sec t+\tan t|)$. Therefore, by Theorem 1, every solution of $y''+y=\tan t$ on $-\pi/2<t<\pi/2$ has the form

$$\psi(t)=c_1\cos t+c_2\sin t-\cos t\log|\sec t+\tan t|$$

for some unique choice of the constants c_1 and c_2.

Exercises

4. Find the general solution of each of the following differential equations.
   a) $y''+y=\sec t, \quad -\pi/2<t<\pi/2$
   b) $y''+4y'+4y=\cos 2t$
   c) $y''+4y=f(t)$, where f is any continuous function on some interval I
   d) $y''-4y'+4y=3e^{-t}+2t^2+\sin t$
   e) $y''+(1/4t^2)y=f(t)$, $(t>0)$, f continuous, given that $\phi_1(t)=\sqrt{t}$ is a solution of the homogeneous equation.
5. Given that ϕ is a solution of the equation $y''+k^2y=f(t)$, where k is a real constant different from zero and f is continuous for $0\le t<\infty$, show that c_1 and c_2 can be chosen so that

$$\phi(t)=c_1\cos kt+\frac{c_2}{k}\sin kt+\frac{1}{k}\int_0^t \sin k(t-s)\, f(s)\,ds$$

for $0\le t<\infty$. (Use $\cos kt$ and $\sin kt/k$ as a fundamental set of solutions of the homogeneous equation.) Find an analogous formula in the case $k=0$.

6. Given the equation

$$y''+5y'+4y=f(t),$$

use the variation-of-constants formula and Theorem 1 to prove that:
   a) If f is bounded on $0\le t<\infty$ (that is, there exists a constant $M\ge 0$ such that $|f(t)|\le M$ on $0\le t<\infty$), then every solution of $y''+5y'+4y=f(t)$ is bounded on $0\le t<\infty$.
   b) If also $f(t)\to 0$ as $t\to\infty$, then every solution ϕ of $y''+5y'+4y=f(t)$ satisfies $\phi(t)\to 0$ as $t\to\infty$.

* 7. Can you formulate Exercise 6 for the general equation

$$y''+a_1y'+a_2y=f(t), \qquad a_1 a_2 \text{ constant}$$

with a_1, a_2 suitably restricted?

The method of variation of constants and Theorem 2 are applicable to the nth-order equation $L_n(y)=f$ with coefficients $a_0, a_1, \dots a_n$, f continuous and $a_0(t)\ne 0$ on some interval I *provided one knows n linearly independent solutions $\phi_1, \phi_2, \dots, \phi_n$ of the homogeneous equation $L_n(y)=0$ on I*. Using the second-order case for motivation, we try to find n functions $u_1, u_2, \dots, u_n$, not all constant, so that

$$\psi_p=u_1\phi_1+u_2\phi_2+\cdots+u_n\phi_n$$

will be a solution of $L_n(y)=f$ on I. If (see the second-order case) $u_1'\phi_1+u_2'\phi_2+\dots+u_n'\phi_n=0$ on I, then $\psi_p'=u_1\phi_1'+\cdots+u_n\phi_n'$ on I and if $u_1'\phi_1'+\cdots+u_n'\phi_n'=0$ on I, then $\psi_p''=u_1\phi_1''+\cdots+u_n\phi_n''$. Continuing in this manner we find that if $u_1', u_2', \ldots, u_n'$ are chosen to satisfy the system of linear algebraic equations on I

$$\begin{aligned}
u_1'\phi_1 + u_2'\phi_2 &+ \cdots + u_n'\phi_n = 0\\
u_1'\phi_1' + u_2'\phi_2' &+ \cdots + u_n'\phi_n' = 0\\
&\vdots\\
u_1'\phi_1^{(n-2)} + u_2'\phi_2^{(n-2)} &+ \cdots + u_n'\phi_n^{(n-2)} = 0\\
u_1'\phi_1^{(n-1)} + u_2'\phi_2^{(n-1)} &+ \cdots + u_n'\phi_n^{(n-1)} = \frac{f}{a_0}
\end{aligned} \tag{3.31}$$

then the function

$$\psi_p = u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n$$

will satisfy $L_n(y) = f$ on I.

Note that the determinant of coefficients of the system of equations (3.31) is $W(\phi_1, \phi_2, \ldots, \phi_n)$, which is different from zero for every t in I since $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent solutions of $L_n(y) = 0$ on I.

Exercise

• 8. Verify that the function $\psi_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$ is a solution of $L_3(y) = f$ on I. [*Hint*: Solve (3.31) by Cramer's rule and integrate to find u_1, u_2, u_3.]

Thus the entire problem is reduced to solving the algebraic system (3.31). Since its determinant of coefficients is $W(\phi_1, \phi_2, \ldots, \phi_n)$, and since the solutions $\phi_1, \ldots, \phi_n$ of $L_n(y) = 0$ on I are linearly independent on I, $W(\phi_1, \ldots, \phi_n)(t) \neq 0$ for t on I and (3.31) always has a unique solution for the quantities $u_1', \ldots, u_n'$ on I. In fact, letting $W_j(t)$ be the n by n determinant having the same elements as $W(\phi_1, \ldots, \phi_n)(t)$ except with $(0, 0, \ldots, 1)$ as its jth column, we see that Cramer's rule (Appendix 1) gives

$$u_j'(t) = \frac{W_j(t)}{W(\phi_1, \ldots, \phi_n)} \frac{f(t)}{a_0(t)}, \qquad j = 1, \ldots, n.$$

The u_j are obtained by integration, so that

$$\psi_p(t) = \sum_{j=1}^{n} \phi_j(t) \int_{t_0}^{t} \frac{W_j(s)}{W(\phi_1, \ldots, \phi_n)(s)} \frac{f(s)}{a_0(s)}\, ds, \tag{3.32}$$

where t_0 and t are any two points of I.

We have therefore sketched the derivation of the following result, which generalizes Theorem 3.

Theorem 4. *Let $\phi_1, \phi_2, \ldots, \phi_n$ be n linearly independent solutions of the equation*

CHAPTER 4

Linear Systems of Differential Equations

In this chapter we will study the theory of systems of linear differential equations, together with an outline of the basic theory of nonlinear systems at the end of the chapter. As we shall see presently, mathematical models of physical systems somewhat more complicated than those investigated in Chapters 1, 2, 3 (motion of several interacting particles, electrical networks, population problems involving more than one species, etc.) often lead to systems of more than one differential equation. Such systems can be reduced in most instances to *linear systems* of first-order differential equations with the aid of certain simplifying assumptions. To study these linear systems, we shall make use of linear algebra (vector spaces and matrix algebra). As a very special case of every result obtained in this chapter we will obtain a corresponding result for a scalar linear differential equation of second (or higher) order, such as those studied in Chapter 3. *Thus students familiar with linear algebra can study this development directly and omit Chapter 3.*

4.1 INTRODUCTION

We shall consider *systems of first-order linear differential equations* of the form

$$\begin{aligned}
y_1' &= a_{11}(t)\, y_1 + a_{12}(t)\, y_2 + \cdots + a_{1n}(t)\, y_n + g_1(t) \\
y_2' &= a_{21}(t)\, y_1 + a_{22}(t)\, y_2 + \cdots + a_{2n}(t)\, y_n + g_2(t) \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad\qquad\ \ \vdots \qquad\quad \vdots \\
y_n' &= a_{n1}(t)\, y_1 + a_{n2}(t)\, y_2 + \cdots + a_{nn}(t)\, y_n + g_n(t),
\end{aligned} \tag{4.1}$$

where the given functions $a_{ij}(t)$, where $i, j = 1, \ldots, n$, and $g_i(t)$, where $i = 1, \ldots, n$, are continuous on some fixed interval $\mathscr{I}$. Unless mentioned specifically otherwise the interval $\mathscr{I}$ can be open, closed, half open, finite or infinite. If $n=1$, we have the important special case of a scalar first-order equation, which we write in the form

$$y' = p(t)\, y + q(t), \tag{4.2}$$

where $p(t)$ and $q(t)$ are given functions.

System (4.1) is linear in $y_1, y_2, \ldots, y_n$, and $y_1', y_2', \ldots, y_n'$. The scalar equation $y' = 2y^2$ is an example of a nonlinear differential equation. These much more complicated equations will not be considered in this chapter; of course some simple cases of nonlinear equations have been studied in Chapter 1.

Example 1. Consider the system

$$\begin{aligned} y_1' &= y_1 - t y_2 + e^t \\ y_2' &= t^2 y_1 - y_3 \\ y_3' &= y_1 + y_2 - y_3 + 2e^{-t}, \end{aligned} \tag{4.3}$$

where $\mathscr{I}$ is the real line, $\{t \mid -\infty < t < \infty\}$. Here, $n=3$, and in the notation of (6.1)

$$\begin{array}{lll|l} a_{11}(t)=1 & a_{12}(t)=-t & a_{13}(t)=0 & g_1(t)=e^t \\ a_{21}(t)=t^2 & a_{22}(t)=0 & a_{23}(t)=-1 & g_2(t)=0 \\ a_{31}(t)=1 & a_{32}(t)=1 & a_{33}(t)=-1 & g_3(t)=2e^{-t}. \end{array}$$

Consider now the array

$$A(t) = \begin{bmatrix} 1 & -t & 0 \\ t^2 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}. \tag{4.4}$$

$A(t)$ is a matrix whose entries are functions. The properties of matrix addition, multiplication by scalars, and matrix multiplication with constant entries also hold for matrices whose entries are functions defined on a common interval $\mathscr{I}$. Let $\mathbf{y}$ and $\mathbf{y}'$ be the column vectors

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix},$$

and let $\mathbf{g}(t)$ be the vector

$$\mathbf{g}(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^{-t} \end{bmatrix}. \tag{4.5}$$

Then, observing that matrix vector multiplication of $A(t)$ and $\mathbf{y}$ gives

$$A(t)\,\mathbf{y} = \begin{bmatrix} y_1 - t y_2 \\ t^2 y_1 - y_3 \\ y_1 + y_2 - y_3 \end{bmatrix}$$

we see that system (4.3) may be represented conveniently in the matrix vector form

$$\mathbf{y}' = A(t)\,\mathbf{y} + \mathbf{g}(t),$$

where $A(t)$ and $\mathbf{g}(t)$ are given respectively by (4.4) and (4.5).

Returning to the general case of system (4.1), we define the $n\times n$ matrix

$$A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \tag{4.6}$$

whose entries are the n^2 functions $a_{ij}(t)$, where $i, j=1, \ldots, n$. Next, define the vectors $\mathbf{g}(t)$, $\mathbf{y}$, $\mathbf{y}'$ by the relations

$$\mathbf{g}(t)=\begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}, \quad \mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{y}'=\begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{bmatrix}. \tag{4.7}$$

Then the system (4.1) can be written in the form

$$\mathbf{y}'=A(t)\,\mathbf{y}+\mathbf{g}(t). \tag{4.8}$$

Exercise

1. Given the system

$$\begin{cases} y'_1=y_2+\cos t \\ y'_2=y_1 \end{cases}$$

Define the matrix $A(t)$ and the vectors $\mathbf{y}$, $\mathbf{y}'$, $\mathbf{g}(t)$, and write this system in the form (4.8).

Before proceeding with the definition of a solution and a discussion of the system (4.8), we need the following definitions.

Definition 1. *A matrix* (such as $A(t)$) *or a vector* (such as $\mathbf{g}(t)$) *is continuous on an interval $\mathscr{I}$ if and only if each of its entries is a continuous function at each point of $\mathscr{I}$.*

Definition 2. *An $n\times n$ matrix $B(t)$ or a vector $\mathbf{u}(t)$ with n components, defined on an interval $\mathscr{I}$ and given respectively by*

$$B(t)=\begin{bmatrix} b_{11}(t) & b_{12}(t) & \cdots & b_{1n}(t) \\ b_{21}(t) & b_{22}(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & & \vdots \\ b_{n1}(t) & b_{n2}(t) & \cdots & b_{nn}(t) \end{bmatrix}, \quad \mathbf{u}(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix},$$

is differentiable on $\mathscr{I}$ if and only if each of its entries is differentiable at every point of $\mathscr{I}$. Their derivatives are given by

$$B'(t)=\begin{bmatrix} b'_{11}(t) & b'_{12}(t) & \cdots & b'_{1n}(t) \\ b'_{21}(t) & b'_{22}(t) & \cdots & b'_{2n}(t) \\ \vdots & \vdots & & \vdots \\ b'_{n1}(t) & b'_{n2}(t) & \cdots & b'_{nn}(t) \end{bmatrix}, \quad \mathbf{u}'(t)=\begin{bmatrix} u'_1(t) \\ u'_2(t) \\ \vdots \\ u'_n(t) \end{bmatrix},$$

respectively. Similarly, the matrix $B(t)$ or the vector $\mathbf{u}(t)$ is integrable on an interval (c, d) if and only if each of its entries is integrable on the interval (c, d). Their integrals are given by

$$\int_c^d B(t)\,dt = \begin{bmatrix} \int_c^d b_{11}(t)\,dt & \int_c^d b_{12}(t)\,dt & \cdots & \int_c^d b_{1n}(t)\,dt \\ \int_c^d b_{21}(t)\,dt & \int_c^d b_{22}(t)\,dt & \cdots & \int_c^d b_{2n}(t)\,dt \\ \vdots & \vdots & & \vdots \\ \int_c^d b_{n1}(t)\,dt & \int_c^d b_{n2}(t)\,dt & \cdots & \int_c^d b_{nn}(t)\,dt \end{bmatrix}$$

$$\int_c^d \mathbf{u}(t)\,dt = \begin{bmatrix} \int_c^d u_1(t)\,dt \\ \int_c^d u_2(t)\,dt \\ \vdots \\ \int_c^d u_n(t)\,dt \end{bmatrix}$$

Exercises

2. Evaluate the derivatives of each of the following vectors or matrices:

a) $B(t)=\begin{bmatrix} t & e^{-t} & 7 \\ \sin t & 0 & \cos t \\ t^2 & t & 1 \end{bmatrix}$ for $-\infty < t < \infty$.

b) $B(t)=\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ for $-\infty < t < \infty$.

c) $B(t)=\begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)\,e^{2t} \end{bmatrix}$ for $-\infty < t < \infty$.

d) $\mathbf{u}(t)=\begin{bmatrix} \log t \\ t\log t \\ t^2 \log t \end{bmatrix}$ for $0 < t < \infty$ (and where $\log t$ is the natural logarithm of t).

e) $\mathbf{u}(t)=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ for $-1<t<2$.

3. Evaluate $\int_1^2 B(t)\,dt$ or $\int_1^2 \mathbf{u}(t)\,dt$ for each of the matrices $B(t)$ or vectors $\mathbf{u}(t)$ in Exercise 1. [*Hint*: In parts (c) and (d), integrate by parts.]
4. Is the vector

$$\mathbf{u}(t)=\begin{bmatrix}1/t\\t^2\end{bmatrix}$$

continuous on the interval $1\le t\le 2$?
Is it continuous on the interval $-1<t<1$? *Explain.*

We are now ready to say what is meant by a solution of system (4.8).

Definition 3. *Let $A(t)$ be a continuous $n\times n$ matrix on an interval $\mathscr{I}$. Let $\mathbf{g}(t)$ be a continuous vector with n components on the same interval $\mathscr{I}$. A solution of the system*

$$\mathbf{y}'=A(t)\,\mathbf{y}+\mathbf{g}(t) \tag{4.8}$$

on some interval $\mathscr{J}$ (where $\mathscr{J}$ is contained in $\mathscr{I}$) *is a vector $\mathbf{u}(t)$ whose derivative $\mathbf{u}'(t)$ is continuous on the interval $\mathscr{J}$ and such that*

$$\mathbf{u}'(t)=A(t)\,\mathbf{u}(t)+\mathbf{g}(t)$$

for every t on $\mathscr{J}$. (Note at this point that the interval $\mathscr{J}$ is not necessarily the same as $\mathscr{I}$.)

Example 2. Consider the scalar ($n=1$) differential equation $y'=-y+1$. Then $u(t)==e^{-t}+1$ is a solution on the interval $-\infty<t<\infty$. For $u'(t)$ is continuous on $-\infty<t<\infty$ and $u'(t)=-e^{-t}$. Thus $u'(t)=-e^{-t}=-u(t)+1$.

Example 3. Show that

$$\boldsymbol{\phi}(t)=\begin{bmatrix}e^t\\e^t\end{bmatrix}$$

is a solution of the system (4.8) for $-\infty<t<\infty$, where $n=2$,

$$A(t)=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \mathbf{g}(t)=\mathbf{0}.$$

Clearly $\phi(t)$ is differentiable for $-\infty<t<\infty$ (because e^t is) and

$$\boldsymbol{\phi}'(t)=\begin{bmatrix}e^t\\e^t\end{bmatrix}.$$

On the other hand, by matrix vector multiplication,

$$A(t)\,\boldsymbol{\phi}(t)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}e^t\\ e^t\end{bmatrix}=\begin{bmatrix}e^t\\ e^t\end{bmatrix}, \qquad -\infty<t<\infty.$$

Thus

$$\boldsymbol{\phi}'(t)=A(t)\,\boldsymbol{\phi}(t), \qquad -\infty<t<\infty.$$

In Chapters 1 and 2, we saw that with a differential equation, one usually associates a particular initial condition. For example, the solution $u(t) = e^{-t}+1$ of Example 2 satisfies at $t=0$ the initial condition $u(0)=2$. More generally, suppose we consider the system (4.8) together with the initial condition $\mathbf{y}(t_0)=\mathbf{y}_0$, where t_0 is a given number in the interval $\mathcal{I}$ and where $\mathbf{y}_0$ is a given vector in n-dimensional Euclidean space.

Definition 4. *By a solution of the initial value problem,*

$$\mathbf{y}'=A(t)\,\mathbf{y}+\mathbf{g}(t), \quad \mathbf{y}(t_0)=\mathbf{y}_0, \tag{4.9}$$

we mean a solution $\mathbf{u}(t)$ *of the system* $\mathbf{y}'=A(t)\,\mathbf{y}+\mathbf{g}(t)$ *on an interval* $\mathcal{I}$ *containing the point* t_0, *such that* $\mathbf{u}(t_0)=\mathbf{y}_0$.

Our object will be to learn as much as possible about such initial value problems. As a matter of fact, when $n=1$ the initial value problem (4.9) can always be solved, and we have already obtained a formula for the solution. (See Section 1.4, Theorem 1.) Unfortunately, for $n\geq 2$ the situation is much more complicated.

Example 4. Show that the vector

$$\mathbf{u}(t)=\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}=\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$

is a solution of the system

$$\mathbf{y}'=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathbf{y}, \quad \text{where } \mathbf{y}=\begin{bmatrix}y_1\\ y_2\end{bmatrix},$$

on $-\infty<t<\infty$, satisfying the initial condition

$$\mathbf{u}(0)=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

Obviously,

$$\mathbf{u}(0)=\begin{bmatrix}\cos 0\\ -\sin 0\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

Since $\cos t$ and $\sin t$ have continuous derivatives everywhere, we have

$$\mathbf{u}'(t)=\begin{bmatrix}-\sin t\\ -\cos t\end{bmatrix}=\begin{bmatrix}u_2(t)\\ -u_1(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathbf{u}(t).$$

Exercises

5. Show that

$$\mathbf{v}(t)=\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

is a solution of the system of Example 4 on $-\infty<t<\infty$ satisfying the initial condition

$$\mathbf{v}(0)=\begin{bmatrix}0\\ 1\end{bmatrix}.$$

6. Show that

$$\mathbf{w}(t)=c_1\mathbf{u}(t)+c_2\mathbf{v}(t),$$

where $\mathbf{u}(t)$, $\mathbf{v}(t)$ are given in Example 5 and Exercise 5, respectively, and where c_1, c_2 are any constants, is a solution of the initial-value problem

$$\mathbf{y}'=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathbf{y},\quad \mathbf{y}(0)=\begin{bmatrix}c_1\\ c_2\end{bmatrix},$$

for $-\infty<t<\infty$.

7. Show that

$$\mathbf{u}(t)=\begin{bmatrix}e^{-t}\\ -e^{-t}\end{bmatrix}$$

is a solution of the initial-value problem

$$\mathbf{y}'=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{y},\quad \mathbf{y}(0)=\begin{bmatrix}1\\ 1\end{bmatrix}$$

on the interval $-\infty<t<\infty$.

If the linear system (4.8) has a very special form, it can be solved completely. We illustrate this with the following examples and exercises.

Example 5. i) Solve the initial-value problem (4.9) with $n=2$,

$$A(t)=\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix},\qquad \mathbf{g}(t)=\mathbf{0}$$

where d_1, d_2 are constants.

Here the system (4.9) has the very simple form

$$\begin{aligned} y_1' &= d_1 y_1, \quad & y_1(t_0) &= y_{01} \\ y_2' &= d_2 y_2, \quad & y_2(t_0) &= y_{02} \end{aligned} \qquad (4.10)$$

in which the differential equations are not linked to one another and each can be solved separately. By separating variables (see, for example, Example 1 and Exercise 1, Section 1.1) we have that $\phi_1(t) = e^{d_1(t-t_0)} y_{01}$ is the solution of the first equation for $-\infty < t < \infty$ and $\phi_2(t) = e^{d_2(t-t_0)} y_{02}$ is the solution of the second equation for $-\infty < t < \infty$. Thus

$$\boldsymbol{\phi}(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} y_{01} \exp d_1(t-t_0) \\ y_{02} \exp d_2(t-t_0) \end{bmatrix}, \qquad -\infty < t < \infty$$

is the solution of (4.10).

ii) Solve the initial-value problem (4.9) where $A(t)$ is the $n \times n$ diagonal matrix

$$A(t) = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \quad \text{and} \quad \mathbf{g}(t) = \mathbf{0}.$$

and where $d_1, d_2, \ldots, d_n$ are constants. It is clear that the jth equation of this system is simply

$$y_j' = d_j y_j, \quad y_j(t_0) = y_{0j}.$$

Its solution (by the same method as in part (i)) is given by

$$\phi_j(t) = y_{0j} \exp d_j(t-t_0), \qquad -\infty < t < \infty.$$

Thus

$$\boldsymbol{\phi}(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_n(t) \end{bmatrix} = \begin{bmatrix} y_{01} \exp d_1(t-t_0) \\ y_{02} \exp d_2(t-t_0) \\ \vdots \\ y_{0n} \exp d_n(t-t_0) \end{bmatrix} \qquad -\infty < t < \infty.$$

iii) Solve the initial-value problem (4.9) if $A(t)$ is the matrix in part (ii) and $\mathbf{g}(t)$ is any continuous vector function $\mathbf{g}(t)$ on $-\infty < t < \infty$. Here the jth equation is

$$y_j' = d_j y_j + g_j(t), \quad y_j(t_0) = y_{0j}$$

which is a linear first-order scalar differential equation. Thus by Theorem 1, Section 1.4, especially Eq. (1.23), with $p(t) = d_j$ and $q(t) = g_j(t)$, we have

$$y_j(t) = y_{0j} \exp d_j(t-t_0) + \int_{t_0}^{t} [\exp d_j(t-s)]\, g_j(s)\, ds$$

and

$$\phi(t)=\begin{bmatrix} y_{01}\exp d_1(t-t_0)+\int_{t_0}^{t}[\exp d_1(t-s)]\,g(s)\,ds \\ y_{02}\exp d_2(t-t_0)+\int_{t_0}^{t}[\exp d_2(t-s)]\,g(s)\,ds \\ \vdots \\ y_{0n}\exp d_n(t-t_0)+\int_{t_0}^{t}[\exp d_n(t-s)]\,g_n(s)\,ds \end{bmatrix}, \qquad -\infty<t<\infty.$$

Exercises

$A=\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}$

8. Find a solution ϕ of the initial-value problem

$$\begin{aligned} y_1' &= -y_1, \\ y_2' &= y_1+y_2, \end{aligned} \qquad \mathbf{y}(0)=\begin{bmatrix}2\\1\end{bmatrix}.$$

[*Hint*: Solve the first equation and substitute in the second equation. Solve this equation by using Theorem 1, Section 14. What is the interval of validity?]

9. Find a solution ϕ of the initial-value problem

$$\begin{aligned} y_1' &= -y_1, \\ y_2' &= y_1+ty_2, \end{aligned} \qquad \mathbf{y}(0)=\begin{bmatrix}1\\1\end{bmatrix}.$$

10. Describe a method for solving the "triangular system"

$$\begin{aligned} y_1' &= a_{11}y_1+a_{12}y_2+\cdots && +a_{1n}y_n \\ y_2' &= \qquad\quad a_{22}y_2+\cdots && +a_{2n}y_n \\ &\vdots && \vdots \\ y_{n-1}' &= && a_{n-1,n-1}y_{n-1}+a_{n-1,n}y_n \\ y_n' &= && a_{nn}y_n, \end{aligned}$$

where a_{ij}, with $j\ge i$, are constants; note that a_{ij}, with $j<i$, are zero.

11. Find a solution of the ϕ of the initial-value problem

$$\begin{aligned} y_1' &= y_1+y_2+f(t), \\ y_2' &= y_1+y_2, \end{aligned} \qquad \mathbf{y}(0)=\begin{bmatrix}0\\0\end{bmatrix},$$

where $f(t)$ is a continuous function. [*Hint*: Define $z(t)=y_1(t)+y_2(t)$.]

Exercises 8, 9, and 10 show that "triangular" systems of first-order differential equations can be solved by successive solution of scalar first-order equations. However, a system that is not triangular cannot, in general, be solved by putting it in triangular form as one might first guess, because the elementary row operations that triangulate the right-hand side (that is, the coefficient matrix A) destroy the isolation of the derivatives on the left-hand side. Note that this difficulty does not arise in the case of linear algebraic equations.

The examples in Section 2.1, particularly Equation (2.11), lead us to a specific linear second-order scalar differential equation with initial conditions. These can be reduced to an initial-value problem for a linear system of two first-order equations of the form (4.9) by the following method.

Example 6. Show that the scalar second-order linear initial value

$$y'' + p(t)\,y' + q(t)\,y = r(t), \qquad y(t_0) = \eta_1, \qquad y'(t_0) = \eta_2, \tag{4.11}$$

where p, q, r are given functions continuous on an interval $\mathscr{I}$, t_0 is in $\mathscr{I}$, and η_1, η_2 are given constants, can be reduced to a system of the form (4.9).

In agreement with Definition 4, by a solution of (4.11) on an interval $\mathscr{J}$ contained in $\mathscr{I}$ we mean a function $w(t)$ such that $w'(t)$, $w''(t)$ exist and are continuous at each point of $\mathscr{J}$, such that $w''(t) + p(t)\,w'(t) + q(t)\,w(t) = r(t)$ for every t in $\mathscr{J}$, and such that $w(t_0) = \eta_1$, $w'(t_0) = \eta_2$. The idea is to introduce new unknowns y_1 and y_2 by means of the definitions $y_1 = y$, $y_2 = y'$. Then

$$\begin{cases} y_1' = y_2 \\ y_2' = y'' = -p(t)\,y' - q(t)\,y + r(t) = -p(t)\,y_2 - q(t)\,y_1 + r(t). \end{cases}$$

This suggests that the given initial-value problem (4.11) can be described by the initial-value problem

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ r(t) \end{bmatrix}, \qquad \mathbf{y}(t_0) = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \boldsymbol{\eta}, \tag{4.12}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Note that (4.12) is a special case of (4.9) with $n = 2$ and $A(t)$, $\mathbf{g}(t)$ displayed in (4.12).

We will now show *the initial-value problems* (4.11) *and* (4.12) *are equivalent in the sense that given a solution of either one, we can construct a solution of the other one.* More precisely, let $\psi(t)$ be a solution of (4.11) on some interval $\mathscr{J}$ containing t_0. Define the functions ϕ_1 and ϕ_2 on $\mathscr{J}$ by the relations

$$\phi_1(t) = \psi(t), \qquad \phi_2(t) = \psi'(t).$$

Define the vector $\boldsymbol{\phi}$ by the relations

$$\boldsymbol{\phi}(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}.$$

We claim that $\boldsymbol{\phi}(t)$ is a solution of (6.13) on $\mathscr{J}$. Clearly,

$$\boldsymbol{\phi}(t_0) = \begin{bmatrix} \phi_1(t_0) \\ \phi_2(t_0) \end{bmatrix} = \begin{bmatrix} \psi(t_0) \\ \psi'(t_0) \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \boldsymbol{\eta};$$

Clearly $\boldsymbol{\phi}(t_0)=\boldsymbol{\eta}$. Moreover, exactly as in Example 5, we have

$$\boldsymbol{\phi}'(t)=\begin{bmatrix}\phi_1'(t)\\ \phi_2'(t)\\ \vdots\\ \phi_{n-1}'(t)\\ \phi_n'(t)\end{bmatrix}=\begin{bmatrix}\psi'(t)\\ \psi''(t)\\ \vdots\\ \psi^{(n-1)}(t)\\ \psi^{(n)}(t)\end{bmatrix}=\begin{bmatrix}\phi_2(t)\\ \phi_3(t)\\ \\ \phi_n(t)\\ -p_1(t)\,\psi^{(n-1)}(t)-\cdots-p_n(t)\,\psi(t)+r(t)\end{bmatrix}$$

$$=\begin{bmatrix}\phi_2(t)\\ \phi_3(t)\\ \vdots\\ \phi_n(t)\\ -p_n(t)\,\phi_1(t)-\cdots-p_1(t)\,\phi_n(t)+r(t)\end{bmatrix}$$

$$=\begin{bmatrix}0 & 1 & 0 & & & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ \vdots & & & & & \\ 0 & 0 & \cdots & & 0 & 1\\ -p_n(t) & & \cdots & & -p_2(t) & -p_1(t)\end{bmatrix}\begin{bmatrix}\phi_1(t)\\ \phi_2(t)\\ \vdots\\ \phi_{n-1}(t)\\ \phi_n(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ r(t)\end{bmatrix}$$

which shows that this particular $\boldsymbol{\phi}$ is a solution of (4.14). Conversely let $\mathbf{u}(t)$ be any solution of (4.14) on $\mathscr{I}$. Define $w(t)=u_1(t)$ (i.e., the first component of u). We claim that this function w is a solution of (4.13) on $\mathscr{I}$. This proof is also very similar to the special case $n=2$ carried out in Example 6 and we shall leave it to the reader as an exercise.

We remark that whereas every nth-order scalar equation is equivalent to a system of first-order equations (as shown in Example 7), the converse is not true. For example, the system

$$\mathbf{y}'=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathbf{y},\quad \mathbf{y}=\begin{bmatrix}y_1\\ y_2\end{bmatrix},$$

cannot be made equivalent to a second-order scalar equation, because the coefficient matrix has the wrong form.

Exercises

12. For each of the following initial-value problems, write an equivalent initial-value problem for a first-order system:

    a) $y''+2y'+7ty=e^{-t}$, $y(1)=7$, $y'(1)=-2$
    b) $2y''-5t^2y'+(\cos t)\,y=\log t$, $y(2)=1$, $y'(2)=0$
    c) $y'''-6y''+3y'+e^{-t}y=\sin t$, $y(0)=0$, $y'(0)=0$, $y''(0)=0$
    d) $y^{(4)}+16y=te^t$, $y(0)=1$, $y'(0)=-1$, $y''(0)=2$, $y'''(0)=0$

13. Reduce each of the following initial-value problems to an equivalent initial-value problem for a first-order system:

a) $y'' + \frac{k}{m} y = \frac{A}{m} \cos \omega t$

$y(0) = y_0, \; y'(0) = 0$

b) $\theta'' + \frac{g}{L} \theta = 0$

$\theta(0) = \theta_0, \; \theta'(0) = 0$

14. Reduce each of the following initial-value problems to an equivalent initial-value problem for a first-order system:

a) $y'' + 5z' - 7y + 6z = e^t$

$z'' - 2z + 13z' - 15y = \cos t,$

where

$$y(0) = 1, \; y'(0) = 0, \; z(0) = 0, \; z'(0) = 1.$$

[*Hint*: Let $w_1 = y$, $w_2 = y'$, $w_3 = z$, $w_4 = z'$.]

b) $y' + 5z + 2y = t^2$

$z'' + 6y' + 11z' - 3y - z = t,$

where

$$y(0) = 1, \; z(0) = 2, \; z'(0) = 3.$$

In Exercise 13 above we have seen how the equations for the simple mass-spring system (Section 2.1) and for the linearized pendulum can be reduced to initial-value problems for linear systems of first-order equations of the form (4.9). To close this introductory section we shall consider two more examples of more complicated physical systems and show how they also lead to an initial-value problem of the form (4.9).

Example 8. A weight of mass m_1 is connected to a rigid wall by a spring having spring constant $k_1 > 0$. A second weight of mass m_2 is connected to the weight of mass m_1 by means of a spring having spring constant $k_2 > 0$. An external face $F(t)$ is applied to the second weight. The whole system slides in a straight line on a frictionless table, as shown in Fig. 4.1. Let $y_1(t)$ denote the displacement of the first weight from its rest position (equilibrium) and $y_2(t)$ the displacement of the second weight from equilibrium. At equilibrium $y_1 = y_2 = 0$ and both springs are unstretched.

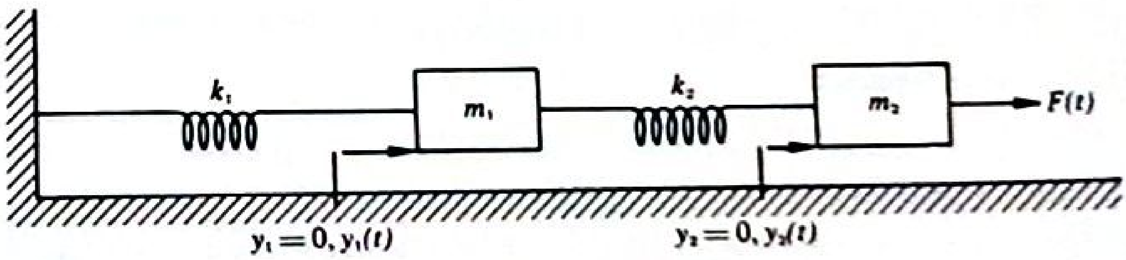

Figure 4.1

a) If at time $t = 0$ the system starts from rest with initial displacements $y_1(0) = y_{10}$, $y_2(0) = y_{20}$, determine the motion of the system.

b) If $m_1 = m_2 = m$, $k_1 = k_2 = k$, and $F(t) \equiv 0$, show that the motion of the system is a superposition of two simple harmonic motions with natural frequencies see that the mathematical problem is to solve the initial-value problem

$$\begin{cases} v_1' = -\frac{1}{3}i_1 + \frac{1}{3}i_s(t), & v_1(0)=0.6 \text{ volt} \\ i_1' = \frac{5}{3}v_1 - \frac{5}{3}v_2, & i_1(0)=1 \text{ ampere} \\ v_2' = 6i_1 - 6v_2, & v_2(0)=1.2 \text{ volts} \end{cases} \tag{4.22}$$

for the unknown functions v_1, i_1, v_2, where $i_s(t)$ is a given source current, that is, a given function defined for $0 \le t < \infty$. This initial-value problem consists of a system of three linear first-order differential equations and it is clearly of the form (4.9). It will be solved completely in Section 5.7.

4.2 THE EXISTENCE AND UNIQUENESS THEOREM

In discussing the scalar first-order differential equation (Theorem 1, Section 1.4), we were able to give an explicit expression for the solution of the initial-value problem. However, for second- or higher-order scalar differential equations and more generally for first-order systems, it is frequently not possible to give an explicit expression for the solution. For example, the scalar differential equation

$$t^2 y'' + t y' + (t^2 - p^2)\, y = 0,$$

where p is a constant, is called the *Bessel equation* (of index p), and arises in many problems of mathematical physics. This equation and its solutions have been studied extensively. Except for special values of p, such as $p=\frac{1}{2}$ or $p=\frac{3}{2}$, these solutions cannot be expressed in terms of a finite number of elementary functions. Nevertheless, the Bessel equation does have solutions for every initial-value problem with $t_0 \neq 0$. This fact is a special case of the following general theorem.

Theorem 1. *Let $A(t)$ be a continuous $n \times n$ matrix on some interval $\mathcal{I}$. Let $\mathbf{g}(t)$ be a vector with n components continuous on the same interval $\mathcal{I}$. Then for every t_0 in $\mathcal{I}$ and every constant vector $\boldsymbol{\eta}$, the initial-value problem*

$$\mathbf{y}' = A(t)\,\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(t_0) = \boldsymbol{\eta} \tag{4.9}$$

has a unique solution existing on the same interval $\mathcal{I}$.

You must refer to Section 8.6 for the proof. Our objective here will be to learn to apply the theorem. In Theorem 1, the matrix $A(t)$ and the vector $\mathbf{g}(t)$ may have real- or complex-valued entries.

Example 1. Let $n=3$ and consider the initial-value problem (4.9) with

$$A(t) = \begin{bmatrix} 1 & -t & 0 \\ \dfrac{1}{t^2-1} & 0 & -1 \\ 2 & \dfrac{1}{t^2+1} & 3 \end{bmatrix}, \quad \mathbf{g}(t) = \begin{bmatrix} e^t \\ \cos t \\ -e^t \end{bmatrix}, \quad t_0 = 0, \quad \boldsymbol{\eta} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Determine whether this initial-value problem has a unique solution and find the largest interval $\mathcal{I}$ of existence of this solution in accordance with the theorem.

The entries of $A(t)$ and $g(t)$, with the exception of $1/(t^2-1)$, are continuous on $-\infty<t<\infty$. However, $1/(t^2-1)$ fails to be continuous at $t=\pm 1$. Since $t_0=0$, Theorem 1, therefore, tells us that the given initial-value problem has a unique solution $\phi(\phi(0)=\eta)$, and the solution ϕ exists on the interval $-1<t<1$. It is worth pointing out that if we choose a different t_0, for example $t_0=10$, the new initial-value problem will also have a unique solution $\psi(\psi(10)=\eta)$ and the solution ψ will exist for $1<t<\infty$.

We now apply Theorem 1 to the important special case of the initial-value problem for a linear second-order scalar equation

$$y''+p(t)\,y'+q(t)\,y=r(t),\quad y(t_0)=\eta_1,\quad y'(t_0)=\eta_2, \tag{4.11}$$

where p, q, and r are functions continuous on some interval $\mathcal{I}$ and t_0 is a point of $\mathcal{I}$. In Example 6, Section 4.1, we have seen that the initial-value problem (4.11) is equivalent to the initial-value problem

$$\mathbf{y}'=\begin{bmatrix}0 & 1\\ -q(t) & -p(t)\end{bmatrix}\mathbf{y}+\begin{bmatrix}0\\ r(t)\end{bmatrix},\quad \mathbf{y}(t_0)=\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} \tag{4.12}$$

to which Theorem 1 can be applied directly. The requirement that $A(t)$ and $g(t)$ be continuous on 1 translates into the following result.

Corollary 1 to Theorem 1. *Let p, q, r be given functions continuous on an interval $\mathcal{I}$ and let t_0 be in $\mathcal{I}$. Then the initial-value problem* (4.11) *has a unique solution $w(w(t_0)=\eta_1,\ w'(t_0)=\eta_2)$ that exists on the same interval $\mathcal{I}$.*

Readers who studied Chapter 3 should note that this is precisely Theorem 1, Section 3.1.

Example 2. Consider the initial-value problem

$$(t^2+4)\,y''+ty'+(\sin t)\,y=1,\quad y(1)=2,\quad y'(1)=0.$$

Determine the existence and uniqueness of the solution as well as the interval of existence.

To apply Corollary 1, we must first reduce the given differential equation to the exact form of (4.11). This is accomplished by dividing by (t^2+4). Thus, in the notation of the corollary

$$p(t)=\frac{t}{t^2+4},\quad q(t)=\frac{\sin t}{t^2+4},\quad r(t)=\frac{1}{t^2+4}.$$

Since these functions are continuous for $-\infty<t<\infty$, the given initial-value problem has a unique solution existing on $-\infty<t<\infty$.

Similar to Corollary 1, we may consider the more general initial-value problem

$$\begin{gathered} y^{(n)}+p_1(t)\,y^{(n-1)}+\cdots+p_{n-1}(t)\,y'+p_n(t)\,y=r(t)\\ y(t_0)=\eta_1,\quad y'(t_0)=\eta_2,\ldots,y^{(n-1)}(t_0)=\eta_n \end{gathered} \tag{4.13}$$

for the scalar nth-order linear equation. From Example 7, Section 4.1, and from Theorem 1 we obtain immediately the following.