Theorem 1. If the complex $n \times n$ matrix $A(t)$ is continuous on an interval $\mathscr{I}$, then the solutions of the system

$$y' = A(t)\, y \tag{4.24}$$

on $\mathscr{I}$ form a vector space V of dimension n over the complex numbers.

In view of the remarks preceding the statement of the theorem, it is significant that, according to Theorem 1, to find any solution of (4.24) it suffices to find a finite number of solutions, namely, a set that forms a basis for the vector space V.

Proof of Theorem 1. We have already established that the solutions form a vector space V over the complex numbers. To establish that the dimension of V is n, we need to construct a basis for V consisting of n linearly independent vectors in V, that is, of n linearly independent solutions of (4.24) on $\mathscr{I}$. We proceed as follows. Let t_0 be any point of $\mathscr{I}$ and let $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_n$ be any n linearly independent points (vectors) in complex Euclidean n-space, (for example, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$,

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow j\text{th row} \\ \\ \\ \\ \end{matrix}$$

are obviously n such vectors.) By Theorem 1, Section 4.2, the system (4.24) possesses n solutions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$, each of which exists on the entire interval $\mathscr{I}$, and each solution $\boldsymbol{\phi}_j$ satisfies the initial condition

$$\boldsymbol{\phi}_j(t_0) = \boldsymbol{\sigma}_j, \quad j = 1, 2, \ldots, n. \tag{4.25}$$

We first show that the solutions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$, are linearly independent on $\mathscr{I}$. Recall that this involves examination of linear combinations of vector functions, but with scalar (constant) coefficients. Suppose there exist complex *constants* $a_1, a_2, \ldots, a_n$, such that

$$a_1\boldsymbol{\phi}_1(t) + a_2\boldsymbol{\phi}_2(t) + \cdots + a_n\boldsymbol{\phi}_n(t) = \mathbf{0} \quad \text{for every } t \text{ on } \mathscr{I}.$$

In particular, putting $t = t_0$, and using the initial conditions (4.25), we have

$$a_1\boldsymbol{\sigma}_1 + a_2\boldsymbol{\sigma}_2 + \cdots + a_n\boldsymbol{\sigma}_n = \mathbf{0}.$$

But this implies that $a_1, a_2, \ldots, a_n$ are all zero because of the assumed linear independence of the given vectors $\sigma_1, \sigma_2, \ldots, \sigma_n$. Thus, $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$ are linearly independent on $\mathscr{I}$.

To complete the proof we must show that these n linearly independent solutions of (4.24) span V; that is, they have the property that any solution $\psi(t)$ of (4.24) can be expressed as a linear combination of the solutions $\phi_1, \phi_2, \dots, \phi_n$. We proceed as follows. Compute the value of the solution ψ at t_0 and let $\psi(t_0)=\sigma$. Since the constant vectors $\sigma_1, \sigma_2, \dots, \sigma_n$ form a basis for complex Euclidean n-space, there exist unique constants $c_1, c_2, \dots, c_n$ such that the constant vector σ can be represented as

$$\sigma = c_1\sigma_1 + c_2\sigma_2 + \cdots + c_n\sigma_n.$$

Now, consider the vector

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t).$$

Clearly, $\phi(t)$ is a solution of (4.24) on $\mathscr{I}$. (Why? Prove this.) Moreover, the initial value of ϕ is (using 4.25)

$$\phi(t_0) = c_1\sigma_1 + c_2\sigma_2 + \cdots + c_n\sigma_n = \sigma.$$

Therefore, $\phi(t)$ and $\psi(t)$ are both solutions of (4.24) on $\mathscr{I}$ with $\phi(t_0)=\psi(t_0) = \sigma$. Thus, by the uniqueness part of Theorem 1, Section 4.2, $\phi(t)=\psi(t)$ for every t on $\mathscr{I}$, and the solution $\psi(t)$ is expressed as the unique linear combination

$$\psi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) \quad \text{for every } t \text{ on } \mathscr{I}. \tag{4.26}$$

Exercise

1. Show that this expression of $\psi(t)$ as a linear combination of $\phi_1(t), \dots, \phi_n(t)$ is unique. [*Hint*: Assume $\psi(t)=d_1\phi_1(t)+\cdots+d_n\phi_n(t)$ in addition to (4.26) and show that $d_j=c_j$, where $j=1,\dots,n$.]

Thus, we have shown that the solutions $\phi_1, \phi_2, \dots, \phi_n$ of (4.24) span the vector space V. Since they are also linearly independent, they form a basis for the solution space V, and the dimension of V is n. This completes the proof of Theorem 1. ∎

We often say that the linearly independent solutions $\phi_1, \dots, \phi_n$ form a *fundamental set of solutions*. There are clearly infinitely many different fundamental sets of solutions of (4.24), namely, one corresponding to every basis $\sigma_1, \dots, \sigma_n$ of Euclidean n-space.

Exercise

2. Prove the following analog of Theorem 1 for systems with real coefficients. If the real $n \times n$ matrix $A(t)$ is continuous on an interval $\mathscr{I}$, then the real solutions of (4.16) on $\mathscr{I}$ form a vector space of dimension n over the real numbers. [*Hint*: This is not a trick question; just check that the proof of Theorem 1 applies here.]

When we wish to apply Theorem 1, it is useful to restate the result in the following manner: *The system* (4.24) *possesses n linearly independent solutions* *on the interval $\mathcal{I}$. Moreover, every solution $\phi(t)$ of* (4.24) *on $\mathcal{I}$ can be expressed as a unique linear combination of those n solutions.* In practice, this means that it suffices to find, in any manner, n solutions of (4.24) and show that they are linearly independent. We shall devote considerable attention to this in special cases later in this chapter. Unfortunately even in special cases this is not a trivial problem, because there does not exist a procedure for finding a basis for the solution space V in the completely general case.

We now apply Theorem 1 to the scalar linear homogeneous second-order differential equation

$$y'' + p(t)\, y' + q(t)\, y = 0, \tag{4.27}$$

where p and q are continuous functions on the interval $\mathcal{I}$.

Corollary 1 to Theorem 1. *Let p and q be continuous on the interval $\mathcal{I}$. Then equation* (4.27) *possesses two linearly independent solutions $\psi_1(t)$, $\psi_2(t)$ on the interval $\mathcal{I}$. Moreover, if $\psi(t)$ is any solution of* (4.27) *on $\mathcal{I}$, then there exist unique constants c_1, c_2 such that*

$$\psi(t) = c_1\psi_1(t) + c_2\psi_2(t) \qquad \text{for every } t \text{ in } \mathcal{I}.$$

Proof. By the method of Example 6, Section 4.1, the scalar equation (4.27) is equivalent to the linear system

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \mathbf{y}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{4.28}$$

which is a special case of (4.24). By Theorem 1, there exist two linearly independent (vector) solutions $\phi_1(t)$, $\phi_2(t)$ of (4.23) such that every solution $\phi(t)$ of (4.28) has the form $\phi(t) = c_1\phi_1(t) + c_2\phi_2(t)$. By the equivalence of (4.27) and (4.28),

$$\phi_1(t) = \begin{bmatrix} \psi_1(t) \\ \psi_1'(t) \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} \psi_2(t) \\ \psi_2'(t) \end{bmatrix},$$

and $\psi_1(t)$, $\psi_2(t)$ are solutions of (4.27) on $\mathcal{I}$. We know that the vector solutions $\phi_1(t)$ and $\phi_2(t)$ are linearly independent on $\mathcal{I}$. We wish to show that $\psi_1(t)$ and $\psi_2(t)$ are linearly independent on $\mathcal{I}$. Suppose that $c_1\psi_1(t) + c_2\psi_2(t) = 0$ for every t in $\mathcal{I}$, then $c_1\psi_1'(t) + c_2\psi_2'(t) = 0$ for every t in $\mathcal{I}$. Thus, $c_1\phi_1(t) + c_2\phi_2(t) = 0$ on $\mathcal{I}$. Since $\phi_1(t)$, $\phi_2(t)$ are linearly independent on $\mathcal{I}$, $c_1 = 0$, $c_2 = 0$. Therefore $\psi_1(t)$ and $\psi_2(t)$ are linearly independent on $\mathcal{I}$. Also, every solution $\psi(t)$ of (4.27) is the first component of the corresponding vector solution $\phi(t)$ of the system (4.28). Since $\phi(t)$ has the form $\phi(t) = c_1\phi_1(t) + c_2\phi_2(t)$, $\psi(t)$ has the form $\psi(t) = c_1\psi_1(t) + c_2\psi_2(t)$. ∎

The same reasoning applied to the scalar linear homogeneous differential equation of order n gives the following result.

Corollary 2 to Theorem 1. *Let $p_1, \ldots, p_n$ be continuous on the interval $\mathscr{I}$. Then the differential equation*

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_n(t)\, y = 0 \tag{4.29}$$

possesses n linearly independent solutions $\psi_1(t), \ldots, \psi_n(t)$ on the interval $\mathscr{I}$. Moreover, if $\psi(t)$ is any solution of (4.29) on $\mathscr{I}$, then there exist unique constants $c_1, \ldots, c_n$ such that $\psi(t) = c_1\psi_1(t) + c_2\psi_2(t) + \cdots + c_n\psi_n(t)$.

Exercise

3. Prove Corollary 2 to Theorem 1.

We can interpret Theorem 1 in a different and useful way. A matrix of n rows whose columns are solutions of (4.24) is called a *solution matrix*. Now, if we form an $n \times n$ matrix using n linearly independent solutions as columns, we will have a solution matrix on $\mathscr{I}$ and also its columns will be linearly independent on $\mathscr{I}$. A solution matrix whose columns are linearly independent on $\mathscr{I}$ is called a *fundamental matrix* for (4.24) on $\mathscr{I}$.... Let us denote the fundamental matrix formed from the solutions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$ as columns by Φ. Then the statement that every solution $\boldsymbol{\psi}$ is the linear combination (4.26) for some unique choice of the constants $c_1, \ldots, c_n$ is simply that

$$\boldsymbol{\psi}(t) = \Phi(t)\, \mathbf{c}, \tag{4.30}$$

where Φ is the fundamental matrix constructed above and $\mathbf{c}$ is the column vector with components $c_1, \ldots, c_n$. (The vector $\Phi(t)\,\mathbf{c}$ is obtained by forming the linear combination of columns of $\Phi(t)$ with $c_1, \ldots, c_n$ as coefficients.) It is clear that if $\tilde{\Phi}$ is any other fundamental matrix of (4.24) in $\mathscr{I}$, then the above solution $\boldsymbol{\psi}$ can be expressed as

$$\boldsymbol{\psi}(t) = \tilde{\Phi}(t)\, \tilde{\mathbf{c}} \quad \text{for every } t \text{ on } \mathscr{I}$$

for a suitably chosen constant vector $\tilde{\mathbf{c}}$. Clearly, every solution of (4.24) on $\mathscr{I}$ can be expressed in this form by using any fundamental matrix.

We see from the discussion above that to find any solution of (4.24) we need to find a fundamental matrix. A natural question, then, is the following. Suppose we have found a solution matrix of (4.24) on some interval $\mathscr{I}$; can we test in some simple way whether this solution matrix is a fundamental matrix? The answer is contained in the following result.

Theorem 2. *A solution matrix $\Phi(t)$ of*

$$\mathbf{y}' = A(t)\, \mathbf{y} \tag{4.24}$$

is a fundamental matrix if and only if $\det \Phi(t) \neq 0$ *for every t in $\mathscr{I}$. Further, if* $\det \Phi(t_0) \neq 0$ *for some t_0 in $\mathscr{I}$, then* $\det \Phi(t) \neq 0$ *for all t in $\mathscr{I}$.* (By $\det \Phi(t)$ we mean the determinant of the matrix $\Phi(t)$.)

Proof. If $\det \Phi(t) \neq 0$ for every t in $\mathscr{I}$, then the columns of the solution matrix $\Phi(t)$ are linearly independent on $\mathscr{I}$. For suppose there exist constants $c_1, \ldots, c_n$ such that

$$c_1\phi_1(t)+c_2\phi_2(t)+\cdots+c_n\phi_n(t)=0 \quad \text{for every } t \text{ in } \mathscr{I},$$

where $\phi_1(t), \ldots, \phi_n(t)$ are the columns of $\Phi(t)$. This can be written in the form

$$\Phi(t)\,\mathbf{c}=\mathbf{0} \quad \text{for every } t \text{ in } \mathscr{I},$$

where

$$\mathbf{c}=\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Fix t at $t=t_0$ in $\mathscr{I}$. Then $\Phi(t_0)\,\mathbf{c}=\mathbf{0}$ is a system of n algebraic equations for the n unknowns $c_1, \ldots, c_n$. Since $\det \Phi(t_0)\neq 0$, $c_1=0$, $c_2=0, \ldots, c_n=0$ by Cramer's rule. This proves that the columns

$$\phi_1(t), \ldots, \phi_n(t)$$

are linearly independent; hence $\Phi(t)$ is a fundamental matrix on $\mathscr{I}$.

Conversely, suppose $\Phi(t)$ is a fundamental matrix of (4.24) on $\mathscr{I}$. Let $\phi(t)$ be a solution of (4.24) on $\mathscr{I}$. By Eq. (4.30), there exists a unique vector $\mathbf{c}$ such that $\phi(t)=\Phi(t)\,\mathbf{c}$ for every t in $\mathscr{I}$. Fix t_0 in $\mathscr{I}$; then, in fact, the constant vector $\mathbf{c}$ is uniquely determined by solving the algebraic system $\Phi(t_0)\,\mathbf{x}=\phi(t_0)$. Since this algebraic system has a unique solution for each right-hand side $\phi(t_0)$, the coefficient matrix $\Phi(t)_0$ has rank n. Hence $\Phi(t_0)$ is nonsingular, and therefore, $\det \Phi(t_0)\neq 0$. This is true for each fixed t_0 in $\mathscr{I}$, and therefore $\det \Phi(t)\neq 0$ for each t in $\mathscr{I}$. It may appear that the vector $\mathbf{c}$ depends on the choice of t_0. However, it does not for the following reason. Since $\phi(t)=\Phi(t)\,\mathbf{c}$ for every t in $\mathscr{I}$, if $t_1\neq t_0$ in $\mathscr{I}$, then $\phi(t_1)=\Phi(t_1)\,\mathbf{c}$. Thus, the unique solution of the algebraic system $\Phi(t_1)\,\mathbf{x}=\phi(t_1)$ is the same vector $\mathbf{c}$ obtained as the unique solution of the algebraic system $\Phi(t_0)\,\mathbf{x}=\phi(t_0)$.

Finally, if $\det \Phi(t_0)\neq 0$ for some t_0 in $\mathscr{I}$, let $\sigma_1=\phi_1(t_0), \ldots, \sigma_n=\phi_n(t_0)$. The vectors $\sigma_1, \ldots, \sigma_n$ are linearly independent, and therefore form a basis for Euclidean n-space. We claim that the solutions $\phi_1(t), \ldots, \phi_n(t)$ are linearly independent on $\mathscr{I}$; for if not, there exist scalars $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\phi_1(t)+c_2\phi_2(t)+\cdots+c_n\phi_n(t)\equiv 0 \text{ on } \mathscr{I}.$$

Putting $t=t_0$, we obtain

$$c_1\sigma_1+c_2\sigma_2+\cdots+c_n\sigma_n=0$$

which contradicts the linear independence of $\sigma_1, \ldots, \sigma_n$. Hence, $\Phi(t)$ is a fundamental matrix of (4.24). Therefore, by the second part of the proof, $\det \Phi(t)\neq 0$ for every t in $\mathscr{I}$. ∎

The reader is warned that a matrix may have its determinant identically zero on some interval, although its columns are linearly independent. Indeed, let

$$\Phi(t)=\begin{bmatrix}1 & t & t^2\\ 0 & 2 & t\\ 0 & 0 & 0\end{bmatrix}.$$

Then clearly $\det \Phi(t)=0$, $-\infty<t<\infty$, and yet the columns are linearly independent. This, according to Theorem 2, cannot happen for solutions of (4.24).

Example 1. Show that

$$\Phi(t)=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}$$

is a fundamental matrix for the system

$$\mathbf{y}'=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\mathbf{y}, \quad \text{where} \quad \mathbf{y}=\begin{bmatrix}y_1\\ y_2\end{bmatrix}.$$

We first show that $\Phi(t)$ is a solution matrix. Let $\phi_1(t)$ denote the first column of $\Phi(t)$; then

$$\phi_1'(t)=\begin{bmatrix}e^t\\ 0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^t\\ 0\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\phi_1(t)$$

for $-\infty<t<\infty$. Similarly, if $\phi_2(t)$ denotes the second column of $\Phi(t)$, we have

$$\phi_2'(t)=\begin{bmatrix}(t+1)\,e^t\\ e^t\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}te^t\\ e^t\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\phi_2(t)$$

for $-\infty<t<\infty$. Therefore, $\Phi(t)=[\phi_1(t), \phi_2(t)]$ is a solution matrix for $-\infty<t<\infty$. By Theorem 2, since $\det \Phi(t)=e^{2t}\neq 0$, $\Phi(t)$ is a fundamental matrix for $-\infty<t<\infty$. By Theorem 2 also, it is enough to compute $\det \Phi(t)$ at one point, for instance $t=0$. Since $\Phi(0)=I$, this gives $\det \Phi(0)=1\neq 0$.

Exercises

4. Show, with the aid of Theorem 2, that

$$\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

is a fundamental matrix for the system $\mathbf{y}'=A\mathbf{y}$, where

$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

5. Show, with the aid of Theorem 2, that

$$\begin{bmatrix}\exp(r_1t) & \exp(r_2t)\\ r_1\exp(r_1t) & r_2\exp(r_2t)\end{bmatrix}$$

is a fundamental matrix for the system $y'=Ay$, where

$$A=\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix},$$

and r_1, r_2 are the distinct roots of the quadratic equation $z^2+a_1z+a_2=0$. (We shall learn in Section 5.3, Exercise 1, how to construct this fundamental matrix.)

Corollary 1 to Theorem 2. *If $\Phi(t)$ is a fundamental matrix of $\mathbf{y}'=A(t)\,\mathbf{y}$ on an interval $\mathcal{I}$ and if $C\in\mathcal{F}_{nn}$ is a nonsingular constant matrix, then $\Phi(t)\,C$ is also a fundamental matrix of $\mathbf{y}'=A(t)\,\mathbf{y}$ on $\mathcal{I}$.*

Proof. Let $\Phi(t)=[\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \dots, \boldsymbol{\phi}_n(t)]$. Then the columns of $\Phi(t)\,C$ are linear combinations of the columns of $\Phi(t)$ (by matrix multiplication). Since the columns of $\Phi(t)$ are solutions, $\Phi(t)\,C$ is a solution matrix on $\mathcal{I}$. But $\det \Phi(t)\,C=\det \Phi(t) \det C$. By Theorem 2, $\det \Phi(t)\neq 0$ on $\mathcal{I}$, and since C is nonsingular, $\det C\neq 0$. Thus, $\det \Phi(t)\,C\neq 0$ on $\mathcal{I}$ and, again by Theorem 2, $\Phi(t)\,C$ is a fundamental matrix on $\mathcal{I}$. ∎

Exercise

6. Show that $C\Phi(t)$, where C is a constant matrix and $\Phi(t)$ is a fundamental matrix, need not be a solution matrix $\mathbf{y}'=A(t)\,\mathbf{y}$.

The converse of Corollary 1 is also true.

Corollary 2 to Theorem 2. *If $\Phi(t)$ and $\Psi(t)$ are two fundamental matrices of $\mathbf{y}'=A(t)\,\mathbf{y}$ on $\mathcal{I}$, then there exists a nonsingular constant matrix C such that $\Psi(t)=\Phi(t)\,C$ on $\mathcal{I}$.*

Proof. Letting ψ_j be the jth column of Ψ, we see from (4.30) that $\psi_j=\Phi\mathbf{c}_j$, $j=1,\dots,n$, where $\mathbf{c}_j$ are suitable constant vectors. Therefore, if we define C as the constant matrix whose columns are the vectors $\mathbf{c}_j$, $j=1,\dots,n$, we have at once that $\Psi(t)=\Phi(t)\,C$ for every t on $\mathcal{I}$. Since

$$\det \Psi(t)=\det \Phi(t) \det C$$

and since $\det \Phi$ and $\det \Psi$ are both different from zero on $\mathcal{I}$ (why?), we also have $\det C\neq 0$ so that C is a nonsingular constant matrix. ∎

Exercises

7. a) Show that

$$\Phi(t)=\begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix}$$

is a fundamental matrix for the system $\mathbf{y}' = A(t)\,\mathbf{y}$, where

$$A(t) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{t^2} & \frac{2}{t} \end{bmatrix}$$

on any interval $\mathscr{I}$ not including the origin.

b) Does the fact that $\det \Phi(0) = 0$ contradict Theorem 2?

8. Show that if a real homogeneous system of two first-order equations has a fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{it} & -e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix},$$

then

0

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

is also a fundamental matrix. Can you find another real fundamental matrix? [*Hint*: Let $\Phi(t) = [\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t).]$ Show that $\mathscr{R}\boldsymbol{\phi}_1(t)$ and $\mathscr{R}\boldsymbol{\phi}_2(t)$ are solutions of $\mathbf{y}' = A\mathbf{y}$, A real, where $\mathscr{R}$ is the real part. By the real part of a vector we mean, of course, the real part of each component. A similar result holds for the imaginary parts of $\boldsymbol{\phi}_1(t)$ and $\boldsymbol{\phi}_2(t)$.]

We shall now apply Theorem 2 to the scalar linear homogeneous second-order equation

$$y'' + p(t)\,y' + q(t)\,y = 0, \tag{4.27}$$

where p and q are continuous functions on a given interval $\mathscr{I}$. As we have seen in the proof of Corollary 1 to Theorem 1, (4.27) is equivalent to the system

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \mathbf{y}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{4.28}$$

If $\Phi(t)$ is a solution matrix of (4.28) on $\mathscr{I}$, then $\Phi(t) = [\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t)]$, where

$$\boldsymbol{\phi}_1(t) = \begin{bmatrix} \psi_1(t) \\ \psi_1'(t) \end{bmatrix}, \quad \boldsymbol{\phi}_2(t) = \begin{bmatrix} \psi_2(t) \\ \psi_2'(t) \end{bmatrix}$$

with $\psi_1(t)$, $\psi_2(t)$ solutions of the scalar equation (4.27). By Theorem 2, $\Phi(t)$ is a fundamental matrix of (4.28) on $\mathscr{I}$ if and only if

$$\det \Phi(t) = \det \begin{bmatrix} \psi_1(t) & \psi_2(t) \\ \psi_1'(t) & \psi_2'(t) \end{bmatrix} \neq 0 \quad \text{for } t \text{ in } \mathscr{I}.$$

This determinant is called the *Wronskian* of $\psi_1(t)$ and $\psi_2(t)$. Thus, by the proof of Corollary 1 of Theorem 1, if $\det \Phi(t) \neq 0$, then the solutions $\psi_1(t)$,

$\psi_2(t)$ of the scalar equation (4.27) are linearly independent on $\mathscr{I}$, and every solution of (4.27) can be written as a linear combination of $\psi_1(t)$ and $\psi_2(t)$. This is one-half of the following result.

Corollary 3 to Theorem 2. *Two solutions ψ_1, ψ_2 of (4.27) on $\mathscr{I}$ are linearly independent on $\mathscr{I}$ if and only if their Wronskian*

$$W[\psi_1(t), \psi_2(t)] = \det \begin{bmatrix} \psi_1(t) & \psi_2(t) \\ \psi_1'(t) & \psi_2'(t) \end{bmatrix}$$

is different from zero for all t in $\mathscr{I}$.

Proof. We must still prove that if the solutions ψ_1, ψ_2 are linearly independent on $\mathscr{I}$, then their Wronskian is different from zero for every t in $\mathscr{I}$. Suppose there is at least one $\hat{t}$ on $\mathscr{I}$ such that $W[\psi_1(\hat{t}), \psi_2(\hat{t})] = 0$. (If no such $\hat{t}$ exists, then there is nothing to prove.) Consider the algebraic system

$$\begin{aligned} a_1\psi_1(\hat{t}) + a_2\psi_2(\hat{t}) &= 0 \\ a_1\psi_1'(\hat{t}) + a_2\psi_2'(\hat{t}) &= 0, \end{aligned}$$

for the unknowns a_1, a_2. By the theory of linear homogeneous algebraic equations, this system has a nontrivial solution $\hat{a}_1$, $\hat{a}_2$, where $\hat{a}_1$, $\hat{a}_2$ are not both zero. Consider the function $\psi(t) = \hat{a}_1\psi_1(t) + \hat{a}_2\psi_2(t)$. Since (4.27) is linear, ψ is a solution of (4.27) on $\mathscr{I}$, and $\psi(\hat{t}) = 0$, $\psi'(\hat{t}) = 0$. By Corollary 1 to Theorem 1, Section 4.2, there is only one solution to the initial value problem consisting of (4.27) together with the initial conditions $y(\hat{t}) = 0$, $y'(\hat{t}) = 0$. Since the identically zero function is a solution of this initial value problem, we conclude that $\psi(t) \equiv 0$ on $\mathscr{I}$. Therefore, $\hat{a}_1\psi_1(t) + \hat{a}_2\psi_2(t) = 0$ for every t in $\mathscr{I}$. Since $\hat{a}_1$, $\hat{a}_2$ are not both zero, ψ_1 and ψ_2 are linearly dependent on $\mathscr{I}$. Thus, if ψ_1 and ψ_2 are linearly independent on $\mathscr{I}$, there can be no such $\hat{t}$, and $W[\psi_1(t), \psi_2(t)] \neq 0$ for every t in $\mathscr{I}$. ∎

By Corollary 1 to Theorem 1 every solution of equation (4.27) on $\mathscr{I}$ has the form $c_1\psi_1 + c_2\psi_2$ for some unique choice of the constants c_1, c_2. For this reason a pair of linearly independent solutions, such as ψ_1, ψ_2, of Eq. (4.27) are said to form a *fundamental set of solutions*.

Exercises

9. Show that e^{2t}, e^{-2t} are linearly independent solutions of $y'' - 4y = 0$ on $-\infty < t < \infty$.
10. Show that $e^{-t/2}\cos\sqrt{3}t/2$, $e^{-t/2}\sin\sqrt{3}t/2$ are linearly independent solutions of $y'' + y' + y = 0$ on $-\infty < t < \infty$.
11. Show that e^{-t}, te^{-t} are linearly independent solutions of $y'' + 2y' + y = 0$ on $-\infty < t < \infty$.
12. Show that $\sin t^2$, $\cos t^2$ are linearly independent solutions of $ty'' - y' + 4t^3y = 0$ on $0 < t < \infty$ or $-\infty < t < 0$. Show that $W(\sin t^2, \cos t^2)(0) = 0$. Why does this fact not contradict Corollary 3 of Theorem 2?

13. a) Let ϕ_1, ϕ_2 be any two solutions on some interval $\mathscr{I}$, of $L(y)=a_0(t)\,y''+a_1(t)\,y'+a_2(t)\,y=0$, where a_0, a_1, a_2 are continuous on $\mathscr{I}$ and $a_0(t)\neq 0$ on $\mathscr{I}$. Show that the Wronskian $W(\phi_1, \phi_2)(t)$ satisfies the first-order linear differential equation

$$W' = -\frac{a_1(t)}{a_0(t)}\,W, \qquad t \text{ on } \mathscr{I}. \tag{*}$$

[*Hint*:

$$W'(\phi_1, \phi_2)(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix}' = (\phi_1\phi_2' - \phi_1'\phi_2)' = \phi_1\phi_2'' - \phi_1''\phi_2.$$

Now, use the fact that ϕ_1, ϕ_2 are solutions of $L(y)=0$ on $\mathscr{I}$ to replace ϕ_1'' and ϕ_2'' by terms involving $\phi_1, \phi_1', \phi_2, \phi_2'$. If you then collect terms you should get Eq. (*).]

b) By solving (*), derive *Abel's formula*:

$$W(\phi_1, \phi_2)(t) = W(\phi_1, \phi_2)(t_0)\exp\left(-\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}\,ds\right),$$

for t_0, t on $\mathscr{I}$. This gives another way of seeing that if the Wronskian is different from zero at one point, then it is never zero.

14. State the analog of Corollary 3 to Theorem 2 for the linear third-order differential equation

$$L_3(y) = a_0(t)\,y''' + a_1(t)\,y'' + a_2(t)\,y' + a_3(t)\,y = 0.$$

15. Show that e^t, $\cos t$, $\sin t$ are linearly independent solutions on $-\infty<t<\infty$ of the differential equation $y'''-y''+y'-y=0$.

16. Show that

$$\phi_1(t) = 1 + \sum_{m=1}^{\infty}\frac{t^{3m}}{2\cdot 3\cdot 5\cdot 6\cdots(3m-1)(3m)},$$

$$\phi_2(t) = t + \sum_{m=1}^{\infty}\frac{t^{3m+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3m)(3m+1)},$$

are linearly independent solutions of $y''-ty=0$ on the interval $-\infty<t<\infty$. (Here you may assume that it has already been shown that ϕ_1 and ϕ_2 are solutions of $y''-ty=0$, but how could you verify this?)

By a similar argument we can establish the following analog of Corollary 3 to Theorem 2 for the scalar equation of order n.

Corollary 4 to Theorem 2. *A set of n solutions* $\psi_1, \psi_2, \dots, \psi_n$ *on* $\mathscr{I}$ *of*

$$y^{(n)} + p_1(t)\,y^{(n-1)} + \cdots + p_n(t)\,y = 0, \tag{4.29}$$

where $p_1, p_2, \ldots, p_n$ are continuous on $\mathscr{I}$, is linearly independent on $\mathscr{I}$ if and only if the Wronskian

$$W[\psi_1(t), \ldots, \psi_n(t)] = \det \begin{bmatrix} \psi_1(t) & \psi_2(t) & \cdots & \psi_n(t) \\ \psi_1'(t) & \psi_2'(t) & \cdots & \psi_n'(t) \\ \vdots & \vdots & & \vdots \\ \psi_1^{(n-1)}(t) & \psi_2^{(n-1)}(t) & \cdots & \psi_n^{(n-1)}(t) \end{bmatrix}$$

is different from zero for every t on $\mathscr{I}$.

A set of n linearly independent solutions of Eq. (4.29) is said to form a *fundamental set of solutions* (see the remarks following Corollary 3 to Theorem 2 in the second-order case).

Exercises

17. Prove Corollary 4 to Theorem 2. [*Hint*: Imitate the proof of Corollary 3 to Theorem 2.]

18. In each of the following, let $\phi_1(t)$ and $\phi_2(t)$ be solutions of the differential equation

$$L(y) = y'' + p(t)\,y' + q(t)\,y = 0$$

on some interval $\mathscr{I}$, where p and q are continuous on $\mathscr{I}$.

a) If $\phi_1(t_0) = \phi_2(t_0) = 0$ for some t_0 in $\mathscr{I}$, show that the solutions ϕ_1 and ϕ_2 cannot form a fundamental set of solutions on $\mathscr{I}$.

b) If the solutions ϕ_1 and ϕ_2 both have a maximum or minimum at some point t_1 in $\mathscr{I}$, show that ϕ_1 and ϕ_2 cannot form a fundamental set of solutions on $\mathscr{I}$.

c) Let ϕ_1 and ϕ_2 form a fundamental set of solutions on $\mathscr{I}$ which both have an inflection point at some point t_2 in $\mathscr{I}$. Show that $p(t_2) = q(t_2) = 0$.

d) Let ϕ_1 and ϕ_2 form a fundamental set of solutions on $\mathscr{I}$. Show that $\psi_1 = \phi_1 + \phi_2$, $\psi_2 = \phi_2 - 2\phi_2$ also form a fundamental set of solutions on $\mathscr{I}$.

19. a) Let ϕ_1 and ϕ_2 be solutions of $L(y) = y'' - 4ty' + (4t^2 - 2)\,y = 0$ on the interval $-\infty < t < \infty$, satisfying the initial conditions $\phi_1(1) = 1$, $\phi_1'(1) = \frac{1}{3}$, $\phi_2(1) = 3$, $\phi_2'(1) = 1$. Are these solutions linearly independent on $-\infty < t < \infty$? Justify your answer.

b) Show that $\psi_1(t) = \exp(t^2)$ is a solution of the equation $L(y) = 0$ and find a second linearly independent solution on $-\infty < t < \infty$. [*Hint*: Look for a solution ψ_2 of the form $\psi_2(t) = u(t)\,\psi_1(t)$; substitute and find $u(t)$ to make ψ_1, ψ_2 linearly independent solutions.]

c) Find the solutions ϕ_1 and ϕ_2 in part (a).

20. Given that the equation

$$ty'' - (2t+1)\,y' + 2y = 0, \qquad t > 0$$

has a solution of the form e^{ct} for some c, find the general solution. [*Hint*: First find what c must be; then find a second linearly independent solution as in Exercise 19(b).]

21. a) One solution of the equation

$$L(y)=t^2y''+ty'+\left(t^2-\frac{1}{3}\right)y=0,\quad t>0$$

is $t^{-1/2}\sin t$. Find the general solution of the equation $L(y)=3t^{1/2}\sin t$, where $t>0$. [*Hint*: Use the method suggested in Exercise 19(b) previously.]

b) Repeat part (a) for the equation

$$2ty''+(1-4t)\,y'+(2t-1)\,y=e^t.$$

To provide the reader with some easy, concrete examples, we advise that he study the solution of linear scalar differential equations with constant coefficients as carried out in Sections 3.4 and 3.5. While this material is an easy special case of linear systems with constant coefficients, to be studied in Chapter 5, it is, nevertheless, helpful to see the special case first.

4.4 LINEAR NONHOMOGENEOUS SYSTEMS

We now use the theory developed in Sections 4.2 and 4.3 to discuss the form of solutions of the nonhomogeneous system

$$\mathbf{y}'=A(t)\,\mathbf{y}+\mathbf{g}(t), \tag{4.31}$$

where $A(t)$ is a given continuous matrix and $\mathbf{g}(t)$ is a given continuous vector on an interval $\mathscr{I}$. *The entire development rests on the assumption that we can find a fundamental matrix of the corresponding homogeneous system* $\mathbf{y}'=A(t)\,\mathbf{y}$. The vector $\mathbf{g}(t)$ is usually referred to as a forcing term because if (4.31) describes a physical system, $\mathbf{g}(t)$ represents an external force. By Theorem 1, Section 4.2, we know that given any point $(t_0, \boldsymbol{\eta})$, t_0 in $\mathscr{I}$, there is a unique solution $\boldsymbol{\phi}$ of (4.31) existing in all of $\mathscr{I}$ such that $\boldsymbol{\phi}(t_0)=\boldsymbol{\eta}$.

To construct solutions of (4.31), we let $\Phi(t)$ be a fundamental matrix of the homogeneous system $\mathbf{y}'=A(t)\,\mathbf{y}$ on $\mathscr{I}$; Φ exists as a consequence of Theorem 1, Section 4.3 (see also remarks immediately following its proof). Suppose $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are any solutions of (4.31) on $\mathscr{I}$. Then $\boldsymbol{\phi}_1-\boldsymbol{\phi}_2$ is a solution of the homogeneous system on $\mathscr{I}$.

Exercise

1. Verify this fact.

By Theorem 1, Section 4.3, and the remarks immediately following its proof (in particular, see Eq. 4.30), there exists a contant vector $\mathbf{c}$ such that

$$\boldsymbol{\phi}_1-\boldsymbol{\phi}_2=\Phi\mathbf{c}. \tag{4.32}$$

Formula (4.32) tells us that to find any solution of (4.31), we need only know one solution of (4.31). (Every other solution differs from the known one by some solution of the homogeneous system.) There is a simple method, known as variation of constants, to determine a solution of (4.31) *provided we know a fundamental matrix for the homogeneous system* $\mathbf{y}'=(At)\,\mathbf{y}$. Let Φ be such a fundamental matrix on $\mathscr{I}$. We attempt to find a solution $\boldsymbol{\psi}$ of (4.31) of the form

$$\boldsymbol{\psi}(t)=\Phi(t)\,\mathbf{v}(t), \tag{4.33}$$

where $\mathbf{v}$ is a vector to be determined. (Note that if $\mathbf{v}$ is a constant vector, then $\boldsymbol{\psi}$ satisfies the homogeneous system and thus for the present purpose $\mathbf{v}(t)\equiv\mathbf{c}$ is ruled out.) Suppose such a solution exists. Then substituting (4.33) into (4.31), we find for all t on $\mathscr{I}$

$$\boldsymbol{\psi}'(t)=\Phi'(t)\,\mathbf{v}(t)+\Phi(t)\,\mathbf{v}'(t)=A(t)\,\Phi(t)\,\mathbf{v}(t)+\mathbf{g}(t).$$

Since Φ is a fundamental matrix of the homogeneous system, $\Phi'(t)=A(t)\,\Phi(t)$, and the terms involving $A(t)\,\Phi(t)\,\mathbf{v}(t)$ cancel. Therefore, if $\boldsymbol{\psi}(t)=\Phi(t)\,\mathbf{v}(t)$ is a solution of (4.31), we must determine $\mathbf{v}(t)$ from the relation

$$\Phi(t)\,\mathbf{v}'(t)=\mathbf{g}(t).$$

Since $\Phi(t)$ is nonsingular on $\mathscr{I}$ we can premultiply by $\Phi^{-1}(t)$ and we have, on integrating,

$$\mathbf{v}(t)=\int_{t_0}^{t}\Phi^{-1}(s)\,\mathbf{g}(s)\,ds, \qquad t_0,\, t \text{ on } \mathscr{I}$$

and, therefore, (4.33) becomes

$$\boldsymbol{\psi}(t)=\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\,\mathbf{g}(s)\,ds, \qquad t_0,\, t \text{ on } \mathscr{I}. \tag{4.34}$$

Thus, if (4.31) has a solution $\boldsymbol{\psi}$ of the form (4.33), then $\boldsymbol{\psi}$ is given by (4.34). Conversely, define $\boldsymbol{\psi}$ by (4.34), where Φ is a fundamental matrix of the homogeneous system on $\mathscr{I}$. Then, differentiating (4.34) and using the fundamental theorem of calculus, we have

$$\boldsymbol{\psi}'(t)=\Phi'(t)\int_{t_0}^{t}\Phi^{-1}(s)\,\mathbf{g}(s)\,ds+\Phi(t)\,\Phi^{-1}(t)\,\mathbf{g}(t)$$

$$=A(t)\,\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\,\mathbf{g}(s)\,ds+\mathbf{g}(t),$$

and using (4.34) again,

$$\boldsymbol{\psi}'(t)=A(t)\,\boldsymbol{\psi}(t)+\mathbf{g}(t)$$

for every t on $\mathscr{I}$. Obviously, $\psi(t_0)=0$. Thus, we have proved the *variation of constants formula*:

Theorem 1. *If Φ is a fundamental matrix of $\mathbf{y}'=A(t)\,\mathbf{y}$ on $\mathscr{I}$, then the function*

$$\psi(t)=\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\,\mathbf{g}(s)\,ds$$

is the (unique) *solution of* (4.31) *satisfying the initial condition*

$$\psi(t_0)=\mathbf{0}$$

and valid on $\mathscr{I}$.

Combining Theorem 1 with the remarks made at the beginning of this section, we see that every solution ϕ of (4.31) on $\mathscr{I}$ has the form

$$\phi(t)=\phi_h(t)+\psi(t) \tag{4.35}$$

where ψ is the solution of Eq. (4.31) satisfying the initial condition $\psi(t_0)=\mathbf{0}$, and ϕ_h is the solution of the homogeneous system satisfying the same initial condition at t_0 as ϕ, for example $\phi_h(t_0)=\eta$.

Example 1. Find the solution of the initial-value problem

$$\mathbf{y}'=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\mathbf{y}+\begin{bmatrix}e^{-t}\\ 0\end{bmatrix},\quad \mathbf{y}=\begin{bmatrix}y_1\\ y_2\end{bmatrix},\quad \mathbf{y}(0)=\begin{bmatrix}-1\\ 1\end{bmatrix}.$$

We have seen in Example 1 Section 4.3 that

$$\Phi(t)=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}$$

is a fundamental matrix of the associated homogeneous system on $-\infty<t<\infty$. Taking the inverse of the matrix $\Phi(t)$, we obtain

$$\Phi^{-1}(s)=\frac{\begin{bmatrix}e^s & -se^s\\ 0 & e^s\end{bmatrix}}{e^{2s}}=\begin{bmatrix}1 & -s\\ 0 & 1\end{bmatrix}e^{-s}.$$

Thus, by Theorem 1 the solution ψ satisfying the initial condition

$$\psi(0)=\begin{bmatrix}0\\ 0\end{bmatrix}$$

is

$$\psi(t)=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}\int_0^t e^{-s}\begin{bmatrix}1 & -s\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^{-s}\\ 0\end{bmatrix}ds$$

$$=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}\int_0^t\begin{bmatrix}e^{-2s}\\ 0\end{bmatrix}ds$$

$$=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}\begin{bmatrix}\frac{1}{2}(1-e^{-2t})\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(e^t-e^{-t})\\ 0\end{bmatrix}.$$

Since $\Phi(0)=I$, the solution of the corresponding homogeneous system satisfying the initial condition

$$\mathbf{y}(0)=\begin{bmatrix}-1\\1\end{bmatrix}$$

is

$$\boldsymbol{\phi}_h(t)=\Phi(t)\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}(t-1)\,e^t\\e^t\end{bmatrix}.$$

By (4.35), the desired solution is

$$\boldsymbol{\phi}(t)=\boldsymbol{\phi}_h(t)+\boldsymbol{\psi}(t)=\begin{bmatrix}(t-1)\,e^t\\e^t\end{bmatrix}+\begin{bmatrix}\frac{1}{2}(e^t-e^{-t})\\0\end{bmatrix}$$
$$=\begin{bmatrix}te^t-\frac{1}{2}(e^t+e^{-t})\\e^t\end{bmatrix}.$$

Exercises

2. Consider the system $\mathbf{y}'=A\mathbf{y}+\mathbf{g}(t)$, where

$$A=\begin{bmatrix}2&1\\0&2\end{bmatrix},\quad \mathbf{y}=\begin{bmatrix}y_1\\y_2\end{bmatrix},\quad \mathbf{g}(t)=\begin{bmatrix}\sin t\\\cos t\end{bmatrix}.$$

Verify that

$$\Phi(t)=\begin{bmatrix}e^{2t}&te^{2t}\\0&e^{2t}\end{bmatrix}$$

is a fundamental matrix of $\mathbf{y}'=A\mathbf{y}$. Find that solution $\boldsymbol{\phi}$ of the nonhomogeneous system for which

$$\boldsymbol{\phi}(0)=\begin{bmatrix}1\\-1\end{bmatrix}.$$

3. Find the solution $\boldsymbol{\phi}$ of the system $\mathbf{y}'=A\mathbf{y}+\mathbf{g}(t)$ with A the same as in Exercise 2 and with

$$\mathbf{g}(t)=\begin{bmatrix}0\\e^{2t}\end{bmatrix},$$

satisfying the initial condition

$$\boldsymbol{\phi}(0)=\begin{bmatrix}1\\-1\end{bmatrix}.$$

4. Consider the system $\mathbf{y}'=A(t)\,\mathbf{y}+\mathbf{g}(t)$, where

$$A(t)=\begin{bmatrix}0&1\\\dfrac{-2}{t^2}&\dfrac{2}{t}\end{bmatrix}\quad \mathbf{g}(t)=\begin{bmatrix}t^4\\t^3\end{bmatrix}.$$

Find the solution $\boldsymbol{\phi}$ satisfying the initial condition

$$\boldsymbol{\phi}(2)=\begin{bmatrix}1\\4\end{bmatrix}$$

and determine the interval of validity of this solution. [*Hint*: Use the fundamental matrix given in Exercise 7, Section 4.3.]

We now consider the form of the variation of constants formula for the scalar second-order linear nonhomogeneous differential equation

$$y''+p(t)\,y'+q(t)\,y=r(t), \tag{4.36}$$

where p, q, and r are continuous on an interval $\mathscr{I}$. We have seen in Corollary 1 to Theorem 1, Section 4.3, that the corresponding homogeneous equation

$$y''+p(t)\,y'+q(t)\,y=0 \tag{4.27}$$

has two linearly independent solutions ψ_1, ψ_2 on $\mathscr{I}$. These solutions are the first components of the (vector) solutions,

$$\boldsymbol{\phi}_1(t)=\begin{bmatrix}\psi_1(t)\\ \psi_1'(t)\end{bmatrix},\quad \boldsymbol{\phi}_2(t)=\begin{bmatrix}\psi_2(t)\\ \psi_2'(t)\end{bmatrix}$$

of the equivalent system

$$\mathbf{y}'=\begin{bmatrix}0 & 1\\ -q(t) & -p(t)\end{bmatrix}\mathbf{y},\quad \mathbf{y}=\begin{bmatrix}y_1\\ y_2\end{bmatrix}. \tag{4.28}$$

We apply Theorem 1 to the system

$$\mathbf{y}'=\begin{bmatrix}0 & 1\\ -q(t) & -p(t)\end{bmatrix}\mathbf{y}+\begin{bmatrix}0\\ r(t)\end{bmatrix},\quad \mathbf{y}=\begin{bmatrix}y_1\\ y_2\end{bmatrix} \tag{4.37}$$

which is equivalent to (4.36). Since $\psi_1(t)$, $\psi_2(t)$ are linearly independent solutions of (4.27), by the equivalence of the equations (4.27) and (4.28), the matrix

$$\Phi(t)=\begin{bmatrix}\psi_1(t) & \psi_2(t)\\ \psi_1'(t) & \psi_2'(t)\end{bmatrix}$$

is a fundamental matrix of the system (4.28) on $\mathscr{I}$. Let

$$\mathbf{u}(t)=\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$

be the solution of (4.37) satisfying the initial condition $\mathbf{u}(t_0)=\mathbf{0}$. By Theorem 1,

$$\mathbf{u}(t)=\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\begin{bmatrix}0\\ r(s)\end{bmatrix}ds.$$

From

$$\Phi^{-1}(s)=\frac{1}{\det\Phi(s)}\begin{bmatrix}\psi_2'(s) & -\psi_2(s)\\ -\psi_1'(s) & \psi_1(s)\end{bmatrix}$$
$$=\frac{1}{W[\psi_1(s),\psi_2(s)]}\begin{bmatrix}\psi_2'(s) & -\psi_2(s)\\ -\psi_1'(s) & \psi_1(s)\end{bmatrix},$$

we obtain

$$\Phi^{-1}(s)\begin{bmatrix}0\\ r(s)\end{bmatrix}=\frac{1}{W[\psi_1(s),\psi_2(s)]}\begin{bmatrix}-\psi_2(s)\,r(s)\\ \psi_1(s)\,r(s)\end{bmatrix},$$

and therefore

$$\mathbf{u}(t)=\Phi(t)\int_{t_0}^{t}\frac{1}{W[\psi_1(s),\psi_2(s)]}\begin{bmatrix}-\psi_2(s)\,r(s)\\ \psi_1(s)\,r(s)\end{bmatrix}ds$$

$$=\int_{t_0}^{t}\frac{1}{W[\psi_1(s),\psi_2(s)]}\begin{bmatrix}\psi_1(t) & \psi_2(t)\\ \psi_1'(t) & \psi_2'(t)\end{bmatrix}\begin{bmatrix}-\psi_2(s)\,r(s)\\ \psi_1(s)\,r(s)\end{bmatrix}ds$$

$$=\int_{t_0}^{t}\frac{1}{W[\psi_1(s),\psi_2(s)]}\begin{bmatrix}\psi_2(t)\,\psi_1(s)-\psi_1(t)\,\psi_2(s)\\ \psi_2'(t)\,\psi_1(s)-\psi_1'(t)\,\psi_2(s)\end{bmatrix}r(s)\,ds.$$

The solution of (4.36) satisfying the initial conditions $y(t_0)=0$, $y'(t_0)=0$ is, by the equivalence of (4.36) and (4.37), the first component $u_1(t)$ of $\mathbf{u}(t)$. Therefore this solution is

$$u_1(t)=\int_{t_0}^{t}\frac{[\psi_2(t)\,\psi_1(s)-\psi_1(t)\,\psi_2(s)]\,r(s)}{W[\psi_1(s),\psi_2(s)]}\,ds.$$

Thus we have proved the variation of constants formula for the scalar second-order linear equation:

Corollary to Theorem 1. *Let $\psi_1(t)$, $\psi_2(t)$ be linearly independent solutions of*

$$y''+p(t)\,y'+q(t)\,y=0 \tag{4.25}$$

on an interval $\mathscr{I}$. Then the function

$$u_1(t)=\int_{t_0}^{t}\frac{[\psi_2(t)\,\psi_1(s)-\psi_1(t)\,\psi_2(s)]\,r(s)}{W[\psi_1(s),\psi_2(s)]}\,ds \tag{4.38}$$

is the unique *solution of*

$$y''+p(t)\,y'+q(t)\,y=r(t) \tag{4.36}$$

on $\mathscr{I}$ satisfying the initial conditions $y(t_0)=0$, $y'(t_0)=0$. Moreover, every solution of (4.36) on $\mathscr{I}$ has the form

$$w(t)=c_1\psi_1(t)+c_2\psi_2(t)+u_1(t) \tag{4.39}$$

for some unique choice of the constants c_1, c_2.

This last expression (4.39) is called the *general solution* of (4.36).

Example 2. Find a particular solution of the differential equation

$$y''+y=\tan t, \qquad -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

We apply the corollary to Theorem 1 directly using the linearly independent solutions $\psi_1(t)=\cos t$, $\psi_2(t)=\sin t$ of the homogeneous equation $y''+y=0$. We have

$$W(\psi_1(t),\psi_2(t))=\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}\equiv 1.$$

Hence formula (4.38) yields (using $t_0=0$) the particular solution

$$u_1(t)=\int_0^t (\sin t\cos s-\cos t\sin s)\tan s\,ds$$

$$=\sin t\int_0^t \sin s\,ds-\cos t\int_0^t \sin s\tan s\,ds$$

$$=\sin t(1-\cos t)+\cos t\int_0^t(\cos s-\sec s)\,ds$$

$$=\sin t(1-\cos t)+\cos t(\sin t-\log|\sec t+\tan t|).$$
$$=\sin t-\cos t\log|\sec t+\tan t|.$$

We note that, since $\sin t$ is a solution of the homogeneous equation, the function

$$u(t)=-\cos t\log|\sec t+\tan t|$$

is also a particular solution. We also remark that we could apply Theorem 1 directly by first converting the given differential equation to an equivalent system of first-order equations as was done for Eq. (4.36); however, for second-order scalar equations, it is more efficient to employ the Corollary.

Exercises

5. Verify by direct substitution that $u_1(t)$ is a solution of (6.36) on $\mathscr{I}$ satisfying the initial conditions $u_1(t_0)=0$, $u_1(t_0)=0$.
6. Find the general solution of each of the following differential equations.

the amplitude B of the oscillation is a maximum if $k=(q-p^2/2)^{1/2}$, called the resonant frequency, provided $p^2<2q$. What happens in the case $p^2 \geq 2q$? Show that at resonance the amplitude of the oscillation is inversely proportional to the damping p. [*Hint*: For the corresponding homogeneous equation, see Exercise 14, Section 3.4.]

4.5 NONLINEAR SYSTEMS OF FIRST-ORDER EQUATIONS

Nonlinear systems of first-order differential equations can be treated in a way which has many points of similarity with the treatment given in Section 4.1 and 4.2 for linear systems. While the algebraic structure of solution sets does not carry over to nonlinear systems, much of the notation, basic existence theory, and reduction of scalar equations of order n to systems is much the same as in the linear case. In this section we shall study systems of first-order differential equations of the form

$$\begin{aligned} y_1' &= f_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= f_2(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, y_2, \ldots, y_n) \end{aligned} \tag{4.40}$$

where $f_1, f_2, \ldots, f_n$ are n given functions defined in some region D of $(n+1)$-dimensional Euclidean space and $y_1, y_2, \ldots, y_n$ are the n unknown functions. *To solve* (4.40) means to find an interval I on the t axis and n functions $\phi_1, \ldots, \phi_n$ defined on I such that

i) $\phi_1'(t), \phi_2'(t), \ldots, \phi_n'(t)$ exist for each t in I.

ii) the point $(t, \phi_1(t), \ldots, \phi_n(t))$ remains in D for each t in I.

iii) $\phi_j'(t)=f_j(t, \phi_1(t), \phi_2(t), \ldots, \phi_n(t))$ for each t in I $\quad (j=1, \ldots, n)$.

Thus (4.40) is the analog of the single equation $y'=f(t, y)$ studied in Chapter 1. Naturally, the functions f_j may be real or complex-valued. We shall assume the real case unless otherwise stated. While the geometric interpretation is no longer so immediate as in the case $n=1$, a solution of (4.40) (that is, a set of n functions $\phi_1, \ldots, \phi_n$ on an interval I) can be visualized as a curve in the $(n+1)$-dimensional region D, with each point p on the curve given by the coordinates $(t, \phi_1(t), \ldots, \phi_n(t))$ and with $\phi_i'(t)$ being the component of the tangent vector to the curve in the direction y_i. This interpretation reduces to the one already given when $n=1$ and the curve in D defined by any solution of (4.40) can therefore again be called a *solution curve*. *The initial-value problem* associated with a system such as (4.40) is the problem of finding a solution (in the sense defined above) passing through a given point P_0: $(t_0, \eta_1, \eta_2, \ldots, \eta_n)$ (we do not write $(t_0, y_{10}, \ldots, y_{n0})$ to avoid double subscripts) of D. In general, we cannot expect to be able to solve (4.40) except in very special cases. Nevertheless, it is desired to obtain as much information as possible about the behavior of solutions of systems. For this reason we shall develop a considerable amount of theory for systems of differential equations.

Example 1. Consider the second-order equation

$$y'' = g(t, y, y') \tag{4.41}$$

where g is a given function. Put $y = y_1$, $y' = y_2$; then one has $y_1' = y_2$ and from (4.41) $y'' = y_2' = g(t, y_1, y_2)$. Thus (4.41) is apparently equivalent to the system of two first-order equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= g(t, y_1, y_2) \end{aligned} \tag{4.42}$$

which is a special case of (4.40) with $n = 2$, $f_1(t, y_1, y_2) = y_2$, $f_2(t, y_1, y_2) = g(t, y_1, y_2)$. To see this equivalence let ϕ be a solution of (4.41) on some interval I; then $y_1 = \phi(t)$, $y_2 = \phi'(t)$ is a solution of (4.42) on I. Conversely, let ϕ_1, ϕ_2 be a solution of (4.42) on I, then $y = \phi_1(t)$ (that is, the first component) is a solution of (4.41) on I.

Exercise

1. Write a system of two first-order differential equations equivalent to the second-order equation

$$\theta'' + \frac{g}{L}\sin\theta = 0$$

with initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$, which describes the motion of a simple pendulum (Section 2.2).

Example 2. The scalar equation of nth order

$$y^{(n)} = g(t, y, y', \ldots, y^{(n-1)}) \tag{4.43}$$

can be reduced to a system of n first-order equations by the change of variable $y_1 = y$, $y_2 = y'$, ..., $y_n = y^{(n-1)}$. Then (4.43) is seen to be equivalent to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= g(t, y_1, y_2, \ldots, y_n) \end{aligned} \tag{4.44}$$

another special case of (4.40).

Exercises

2. Establish the equivalence of (4.43) and (4.44).
3. Reduce the system

$$\begin{aligned} y_1' + y_2' &= y_1^2 + y_2^2 \\ 2y_1' + 3y_2' &= 2y_1y_2 \end{aligned}$$

to the form (4.40). [*Hint*: Solve for y_1' and y_2'.]