Linear Transformations or Homomorphism of Linear Spaces.

(Some time also called linear operator)

Def.: Let X and Y be two linear spaces over the same field F. Then a mapping $T: X \to Y$ is called a homomorphism or linear transfomation from X into Y if

(i) $T(x+y) = T(x) + T(y)$, $\forall x, y \in X$,

(ii) $T(\alpha x) = \alpha T(x)$, $\forall \alpha \in F$, $\forall x \in X$.

- (i) & (ii) are equivalent to a single condition

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \alpha, \beta \in F \ \& \ \forall x, y \in X$$

- The T-image of x ($T(x)$) will also written as Tx.

Check 1,2,3.

Ex. 1: The function $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1, x_2) \ \forall x_1, x_2, x_3 \in \mathbb{R}$$

is linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$.

Ex. 2: Eeach of following mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ can be shown to be a linear transfomation on $\mathbb{R}^2$:

$T_1(x_1, x_2) = (\alpha x_1, \alpha x_2)$ where $\alpha \in \mathbb{R}$. $T_2(x_1, x_2) = (x_2, x_1)$.

$T_3(x_1, x_2) = (x_1, 0)$. $T_4(x_1, x_2) = (0, x_2)$.

Ex-3: Let $X = \{ f \mid f: [0,1] \to \mathbb{R}, f \text{ is bounded and cont}^s. \}$.

Then the mapping I defined by

$$I(f) = \int_0^1 f(x)\, dx \ , \ f \in X, \ x \in [0,1].$$

is linear transformation from X into $\mathbb{R}$ (real linear space).

Properties of Linear Transformations:

Theo.: Let $T: X \to Y$ be a linear transformation from X into Y, Then

(a) T preserves the origin i.e., $T(\vec{0}) = 0 \in Y$.

(b) T preserves negative i.e. $T(-x) = -T(x) \ \forall x \in X$.

Proof: (a) $\underset{\in X}{T(0)} = T(\underset{\in F = R}{0}\,\underset{\in X}{0}) = 0\underbrace{T(0)}_{\in Y} = \underset{\in Y}{0}$

(b) $T(-x) = T((-1)x)) = (-1)T(x) = -T(x)$

Some Particular Linear Transformations:

Let X & Y be two linear spaces over the same field F,

1. Zero transformation: $T: X \to Y$ s.t $T(x) = 0 \ \forall x \in X$.
2. Identity transformation: $I: X \to X$ s.t $I(x) = x \ \forall x \in X$.
3. Negative of a linear transformation.
   Let $T: X \to Y$ be a linear transformation.
   then $-T: X \to Y$ s.t $(-T)(x) = -T(x) \ \forall x \in X$
   is linear transformation.

Range and Null space of a Linear transformation:

Def.: Let $T: X \to Y$ be a linear transformation,

(i) Then the range of T written as $R(T)$ is:

$$R(T) = \{T(x) : x \in X\}$$

(ii) then the null space of T written as $N(T)$ is
$$N(T) = \{x \in X : T(x) = 0 \in Y\}.$$

If we regard the linear transformation $T: X \to Y$ as a vector space homomorphism of X into Y, then the null space of T is also called the kernel of T.

Theo.: Let $T: X \to Y$ be a linear transformation, then

(i) $R(T)$ is a subspace of Y.
(ii) $N(T)$ is a subspace of X.

Proof. (check).

(iii) If X is finite dimensional, Then $R(T)$ is a finite-dimensional subspace of Y.

Proof. (iii): If $\dim X = 0$ i.e, $X = \{0\} \Rightarrow R(T) = \{0\}$
i.e., $R(T)$ is of dimension zero.

Now, let $\dim X = n$, where $n > 0$, and let $\{x_1, x_2, \ldots, x_n\}$ be a basis for X.

Let $x' \in R(T)$ be any vector. Then $\exists x \in X$ s.t
$$T(x) = x'.$$
Now, let $x \in X \Rightarrow \exists \alpha_1, \ldots, \alpha_n \in F$ s.t
$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n$$
$$\Rightarrow T(x) = T(\alpha_1 x_1 + \cdots + \alpha_n x_n)$$

$\Rightarrow x' = \alpha_1 T(x_1) + \cdots + \alpha_n T(x_n)$

This mean that, any vector in $R(T)$ can be expressed as a linear combination of $T(x_1), \dots, T(x_n)$ (finite).

Thus $R(T)$ is generated by a finite number of vectors. Hence $R(T)$ is finite dimension.

Rank and Nullity of a Linear transformation.

Def.: Let $T: X \to Y$ be a linear transformation, and let X is finite dimensional. Then
the rank of T (rank(T)) is the dimension of $R(T)$, and
the nullity of T (nullity(T)) is the dimension of $N(T)$.

Theo. Let $T: X \to Y$, with X as finite dimensional. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim X.$$

Linear Transformations As Vectors.

Let X and Y be linear spaces over the same field F.
And Let $Hom(X,Y) = \{T \mid T: X \to Y \text{ be a linear transformation}\}$.

Theo. Let X and Y be linear spaces over the same field F.
Let T_1 & T_2 be linear transformations from X into Y.
Define $T_1 + T_2$ and αT_1, $\alpha \in F$ as follows:

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) \quad \forall x \in X \quad \text{----(1)}$$
$$(\alpha T_1)(x) = \alpha T_1(x) \quad \forall x \in X \quad \text{----(2)}$$

Show that:

(a) The functions $T_1 + T_2$ & αT_1 are linear transformations from X into Y.

(b) $Hom(X,Y)$ with addition in (1) and scalar multiplication in (2) is a linear space over the field F.

Proof. (check):

Def. (Product of Linear Transformations):
Let T and S be two linear transformations on a linear space X, Then the composite function ST (called product of linear transformation) defined as:

$$(ST)(x) = S[T(x)] \ \forall x \in X.$$

Remark: (a) ST is a linear transformation on X. (show that)
(b) In general $TS \neq ST$

Ex. let T_1 & T_2 be linear transformations on $\mathbb{R}^2$ defined as:

$$T_1(x_1, x_2) = (x_2, x_1) \ \& \ T_2(x_1, x_2) = (x_1, 0).$$

Then $T_1 T_2 \neq T_2 T_1$. (check)

Def. (Invertible or Non-Singular linear transformations).
Let T be a linear transformation on a linear space X. Then T is called invertible or non-singular if T is one-one and onto,
otherwise T is called singular.

Suppose $T: X \to X$ be a non-singular linear transformation.

Then T is 1-1 and onto $\Rightarrow$ its inverse T^{-1} exist.

$$T^{-1}: X \to X \quad \text{s.t} \quad T(x) = y \Leftrightarrow T^{-1}(y) = x.$$

If I is identity fun. on X, then $T^{-1}T = I = TT^{-1}$.

<u>Theo.</u> Let T be a non-singular linear transformation on a linear space X. Then

(a) T^{-1} is also a linear transformation on X

(b) $\Leftrightarrow$ $\exists$ linear trans. T' on X s.t $TT' = T'T = I$.

(C) If X finite dimensional, T is non-singular $\Leftrightarrow$ T is 1-1 (or onto).

Def. (Linear Functionals).

Let X be a linear space over the field F.
A function $f: X \to F$ is said to be a linear functional on X if,

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall \alpha, \beta \in F \text{ and } x, y \in X.$$

If f is a linear functional on X (over F), then $f(x) \in F$ $\forall x \in X$
$\therefore$ $f(x)$ is a scalar, then f is a scalar-valued function.

Ex. Let $f: \mathbb{R} \to \mathbb{R}$ be any real valued function, such as:
1. $f(x) = x$. 2. $f(x) = \sin x$ or $\cos x$. 3. $f(x) = e^{x}$. 4. $f(x) = x^2$ or x^3
5. $f(x) = 4x^4 + 3x + 4$.

Then: (a) All above functions are transformations (or operators) from R into R_1.

(b) Eeach one is a functional.

(c) 1. is linear functional, but 2-5 are nonlinear functional.

Continuous Linear Transformations (or Operators).

Let X and Y be two normed spaces with the same scalars. And let $T: X \to Y$ be a linear transformation (or linear operator). Then

(a) T is continuous at a point $x_0 \in X$, if
$$\forall \epsilon > 0 \quad \exists \delta(\epsilon, x_0) \text{ s.t } \|Tx - Tx_0\| < \epsilon \text{ when } \|x - x_0\| < \delta.$$

This is equivalent to saying that,

T is continuous at a point $x_0 \in X$ iff
$$\forall \{x_n\} \text{ sequence in } X \text{ s.t } x_n \to x_0 \Rightarrow Tx_n \to Tx_0.$$

And T is called continuous on X if it is cont.^{s} at every point of X.

(b) T is uniformly continuous if $\forall \epsilon > 0 \ \exists \delta(\epsilon)$ s.t
$$\forall x, y \in X \text{ with } \|x - y\| < \delta \Rightarrow \|Tx - Ty\| < \epsilon.$$

(c) T is called bounded linear transformation if
$$\exists \text{ a real number } K \geq 0 \text{ s.t } \|Tx\| \leq K\|x\| \quad \forall x \in X.$$
(Such K is called a bound for T).