## 1.5. Logical Implication

**Definition 1.5.1. (Logical implication)**

We say the logical proposition **"r"** implies the logical proposition **"s"** (or **s** logically deduced from **r**) and write (**r** $\Longrightarrow$ **s**) iff (**r** $\rightarrow$ **s**) is a tautology.

**Example 1.5.2.** Show that $[(p \rightarrow t) \wedge (t \rightarrow q)] \Longrightarrow (p \rightarrow q)$.

**Solution.** Let P: the proposition $(p \rightarrow t) \wedge (t \rightarrow q)$

Q: the proposition $p \rightarrow q$

| p | t | q | $p \rightarrow t$ | $t \rightarrow q$ | P | Q | $P \rightarrow Q$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | T | F | T | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | F | T | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T |

**Remark 1.5.3.**

**(i)** We use $(r \Longrightarrow s)$ to imply that the statement $(r \rightarrow s)$ is true, while the statement $(r \rightarrow s)$ alone does not imply any particular truth value. The symbol is often used in proofs as shorthand for "**implies**".

**(ii)** If $(r \Longrightarrow s)$ and $(s \Longrightarrow r)$, then we denote that by $(r \Longleftrightarrow s)$.

**Example 1.5.4.** Show that
**(i)** $(r \Longrightarrow s) \equiv [(\sim r \vee s)$ is tautology$]$.
**(ii)** $(r \Longleftrightarrow s) \equiv (r \equiv s)$.

**Solution.**
**(i)**
**1-** $(r \Longrightarrow s) \equiv (r \rightarrow s)$ is tautology (Def. of $\Longrightarrow$)

2- $(r \rightarrow s) \equiv (\sim r \vee s)$ Logical Implication Law
3- $(\sim r \vee s)$ is tautology Inf. (1),(2)

**(ii)**

1- $\left.\begin{array}{l}(r \Rightarrow s) \equiv (r \rightarrow s) \text{ is tautology} \\ (s \Rightarrow r) \equiv (s \rightarrow r) \text{ is tautology}\end{array}\right\}$ Def. of $(\Rightarrow)$ and $(\Leftrightarrow)$

2- $(r \rightarrow s) \wedge (s \rightarrow r)$ is tautology.
3- $r \leftrightarrow s$ is tautology Equivalence Law
4- $r \equiv s$. Inf. Remark 1.3.11

- Generally, the statement and its converse not necessary equivalent. Therefore, $p \Rightarrow q$ does not mean that $q \Rightarrow p$.

**Example 1.5.5.** The statement “the triangle which has equal sides, has two equal legs” equivalent to the statement “ the triangle which has not two equal legs has no equal sides”.

## 1.6. Quantifiers

**Definition 1.6.1.**
**(i)** A **predicate** or **propositional function** is a statement (formula) containing variables and that may be true or false depending on the values of these variables.

- That is, a predicate is a property or relationship between objects represented symbolically.
- We represent a predicate by a letter followed by the variables enclosed between parenthesis: $P(x)$, $Q(x,y)$, etc.

**(ii)** An ***example*** for $P(x)$ is: value of $x$ for which $P(x)$ is true.
**(iii)** A ***counterexample*** $P(x)$ is: value of $x$ for which $P(x)$ is false.
**(iv)** The set, $X$ which contain all possible value that satisfy the formula $P$ is called a **universal set**.
**(v)** A set $Y$ which contains all values $x$ belong to set $X$ such that $P(x)$ is true is called a **solution set.**

$$Y = S_P = \{x \in X : P(x) \text{is true}\}$$

**Example 1.6.2.**

**(i)** $P(x) = x \leq 5 \wedge x > 3$ is true for $x = 4$ and false for $x = 6$ (counterexample).
**(ii)** $P(x) = x \leq 5 \wedge x > 3$, for every real numbers, $x$ which is definitely false.

**(iii)** There exists an $x$ such that $P(x) = x \leq 5 \wedge x > 3$," which is definitely true.

**(iv)** Given the statement "**Ahmad is a logician**".

Let $P$ represent **'is a logician'** and let $x$ represent **'Ahmad'**. The predicate form of this statement is $P(x)$. That is, $P(x) =$ Ahmad is a logician.

**(v)** Let r: $x$ is married to y.

Let $M$ represent **"married".** Then, $\mathrm{r} = M(x,y)$.

**(vi)** Let r: The numbers $x$ and $y$ are both odd.
This statement means ($x$ is odd) $\wedge$ ($y$ is odd).

Let $P$ represent **'is a odd'** and let $x, y$ represent **'numbers'**. The predicate form of this statement is $P(x) \wedge P(y)$.

**Definition 1.6.3.**

**(i)** The phrase **"for all $x$"** (**"for every $x$"**, **"for each $x$"**) is called a **universal quantifier** and is denoted by $\forall\, x$.

**(ii)** The phrase **"for some $x$"** (**"there exists an $x$"**) is called an **existential quantifier** and is denoted by $\exists\, x$.

**Definition 1.6.4. (The Universal Quantifier Proposition)**

Let $f(x)$ be a proposition function which depend only on $x$. A sentence $\forall\, x, f(x)$ read "For all $x$, $P(x)$" mean

"For all values $x$ in $X$(universal set), the predicate $f(x)$ is true."; that is,

$$\frac{\forall x, f(x)}{\therefore\ f(a)}$$

**Example 1.6.5.**

**(i)** r: The square of all real numbers are positive.

$$\boxed{\text{r: } \forall x \in \mathbb{R}, \quad (x^2 \geq 0).}$$

**(ii)** r: The commutative law of addition of real numbers is holed.

$$\boxed{\text{r: } \forall\, x, \forall\, y \in \mathbb{R}, \quad (x + y = y + x).}$$

**(iii)** r: The associative law of addition of real numbers is holed.

$$\boxed{\text{r: } \forall\, x, \forall\, y, \forall\, z \in \mathbb{R}, \ \big((x + y) + z = x + (y + z)\big).}$$

**(iv)** r: All logicians are exceptional.
Let $L$ represent **'set of logician'** and let $E$ represent **'is exceptional'**. The predicate form of this statement is $\boxed{\text{r: } \forall x \in L,\ E(x)}$.

**(v)** r: All cars are red.
Let $X$ ≔ **Set of cars**, $f$: = **is red**. Then, r : $\forall x \in X, f(x)$.

**Remark .1.6.6.**

**(i)** The "**all**" form, the universal quantifier, is frequently encountered in the following context: $\forall x\ (f(x) \longrightarrow Q(x)),$

which may be read,

**"For all $x$ in a universal set $X$ satisfying $f(x)$ also satisfy $Q(x)$".**

For example:

**(a)** r: All logicians are exceptional.

Let $L$ represent **'is a logician'** and let $E$ represent **'is exceptional'**. Then

- Predicate Logic: r: $\forall x(L(x) \longrightarrow E(x))$
- In logical English: "For all $x$, if $x$ is a logician, then $x$ is exceptional."

**(b)** r: The square of all real numbers are positive.
Let $P$ represent: $\in\ \mathbb{R}$ and let $Q$ represent **" is positive"**.

- Predicate Logic: r: $\forall x(P(x) \longrightarrow Q(x))$; that is,
  $\boxed{\text{r: } \forall x(\text{ if } x \in \mathbb{R} \longrightarrow (x^{2} \geq 0).}$
- In logical English: "For all $x$, if $x$ is real number, then $x$ is positive."

**(c)** Every(each, any) integer number is even (or: Integer numbers are even).

Let $P$ represent: $\in\ \mathbb{Z}$ and let $E$ represent **" is even"**.

- Predicate Logic: r: $\forall x(P(x) \longrightarrow E(x))$; that is,
  $\boxed{\text{r: } \forall x(\text{ if } x \in \mathbb{Z} \longrightarrow E(x)).}$
- In logical English: "For all $x$, if $x$ is an integer, then $x$ is even."

**(ii)** Parentheses are crucial here; be sure you understand the difference between the "all" form and $\boxed{\forall x, f(x) \longrightarrow \forall x, Q(x)}$ and $\boxed{(\forall x,\ f(x)) \longrightarrow Q(x)}$.

**Definition 1.6.7. (The Existential Quantifier Proposition)**

A sentence $\exists\ x, f(x)$ read "For some $x$, $P(x)$" or "For some $x$ such that $P(x)$" mean

"For some $x \in X$(universal set), the predicate $f(x)$ is true"; that is,

$$\frac{f(a)}{\therefore \exists x, f(x)}$$

**Example 1.6.8.**

**(i)** $\exists\, x: (x \geq x^2)$ is true since $x = 0$ is a solution. There are many others.

**(ii)** r: Some logicians are exceptional.
Let $L$ represent **'set of logician'** and let $E$ represent **'is exceptional'**. The predicate form of this statement is $\boxed{\text{r: } \exists x \in L,\ E(x)}$.

**(iii)** r: There is a car which is red.
Let $X :=$ **Set of cars**, $f :=$ **is red**. Then, r : $\exists x \in X, f(x)$.

**Remark .1.6.9.**

**(i)** The *"***some***"* form, the existential quantifier, is frequently encountered in the following context: $\exists x\ (f(x) \wedge Q(x))$,

which may be read,

**"Some $\boldsymbol{x}$ in a universal set $\boldsymbol{X}$ satisfying $\boldsymbol{f}(\boldsymbol{x})$ and satisfy $\boldsymbol{Q}(\boldsymbol{x})$"**.

For example:

**(a)** r: Some logicians are exceptional.

Let $L$ represent **'is a logician'** and let $E$ represent **'is exceptional'**. Then

- Predicate Logic: r: $\exists x(L(x) \wedge E(x))$
- In logical English: "For some $x$, $x$ is a logician and $x$ is exceptional."

**(b)** r: The square of some integers numbers are four (or: There is an integer for which its square is four)
Let $P$ represent: $\in \mathbb{Z}$ and let $Q$ represent **" is 4"**.
- Predicate Logic: r: $\exists x(P(x) \wedge Q(x))$; that is,

$$\boxed{\text{r: } \exists x(\, x \in \mathbb{Z} \;\; \wedge \;\; x^{2} = 4).}$$

- In logical English: "For some $x$, $x$ is an integer number and $x^{2} = 4$."

**(c)** At least one integer number is even (or: Some Integers are even).

Let $P$ represent: $\in \mathbb{Z}$ and let $E$ represent **" is even"**.

- Predicate Logic: r: $\exists x(P(x) \wedge E(x))$; that is,

$$\boxed{\text{r: } \exists x(x \in \mathbb{Z} \wedge E(x)).}$$

- In logical English: "For some $x$, $x$ is an integer number and $x$ is even."

**Negation Rules of Quantifiers 1.6.10.**
**(i)** When we negate a quantified statement, we negate all the quantifiers first, from left to right (keeping the same order), then we negative the statement.
**(ii)** $\sim (x = y) = (x \neq y)$.
**(iii)** $\sim (x \equiv y) = (x \not\equiv y)$.
**(iv)** $\sim (x < y) = (y \leq x)$.
**(v)** $\sim (x \in Y) = (x \notin Y)$.
**(vi)** ~ (Even number) = Odd number.

Now define the a formal universal quantifier proposition using negation.

**Definition 1.6.11.**

**(i)** $\forall x, f(x) = \sim\exists\, x, \sim f(x)$.
**(ii)** $\exists\, x, f(x) \equiv \sim\forall x, \sim f(x)$.

**Example 1.6.12.**
r: All logicians are exceptional.

Let $L$ represent **'set of logician'** and let $E$ represent **'is exceptional'**.

- Predicate Logic: $\boxed{\text{r: } \forall x \in L,\;\; E(x) = \sim\exists\, x, \sim E(x)}$.
- In logical English: "There is no $x$ is a logician, for which $x$ is not exceptional."

**Equivalent Definitions 1.6.13.**
**(i)** $\sim\big(\forall x, f(x)\big) \equiv\; \exists\, x, \sim f(x)$.
**(ii)** $\sim\big(\exists\, x, f(x)\big) \equiv \forall\, x, \sim f(x)$.

**(iii)** $\sim \left[\forall x \left(f(x) \longrightarrow Q(x)\right)\right] \equiv \exists x \left(f(x) \wedge \sim Q(x)\right)$
$\equiv$ Some $f(x)$ are not $Q(x)$

**(iv)** $\sim \left(\exists x, \left(f(x) \wedge Q(x)\right)\right) \equiv \forall x, \sim f(x) \vee \sim Q(x) \equiv \forall x\ (f(x) \longrightarrow \sim Q(x))$
$\equiv$ No $f(x)$ are $Q(x)$

**Example 1.6.14.**

**(i)** Express each of the following sentences in the form $\forall x, P(x)$ and then give its negation in both cases $\forall x, P(x)$ and in words.

r: The square of every real number is non-negative.
**Solution.**

- $\boldsymbol{\forall x, P(x)}$ **form:** r: $\forall x \in \mathbb{R}, x^2 \geq 0$.
- **Negation:** ~r: $\sim (\forall x \in \mathbb{R}, x^2 \geq 0) \equiv \exists x \in \mathbb{R}, \sim (x^2 \geq 0) \equiv \exists x \in \mathbb{R}, x^2 < 0$.
- **Negation in words:** ~r:There exists a real number whose square is negative.

**(ii)** Let **r: Student who is intelligent will succeed**. Write “ r ” in predicate logic and English logic, and then give its negation in both cases.
**Solution.**
Let P: Student
Q: intelligent.
S: Succeed

- **Predicate Logic:** **r:** $\forall x((\mathrm{P}(x) \wedge \mathrm{Q}(x)) \rightarrow \mathrm{S}(x))$
- **Negation:** ~r: $\sim \left[\forall x \left(\left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \rightarrow \mathrm{S}(x)\right)\right]$
$\equiv \sim \left[\forall x \left(\sim \left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \vee \mathrm{S}(x)\right)\right]$ Implication Low.
$\equiv \exists x \left(\left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \wedge \sim \mathrm{S}(x)\right)$ De Mover’s Law.
- **English logic:** ~r: There exist student who is intelligent and not succeed.

**(iii)** r: Some integer numbers are even but not odd.

Let $\mathbb{Z} \coloneqq$ **Set of Integers**, $f := $ **is even**, $P := $ **is odd**.

- **Predicate Logic:** r : $\exists x \in \mathbb{Z}, \left(f(x) \wedge \sim P(x)\right) \equiv \sim \left[\forall x \left(f(x) \longrightarrow P(x)\right)\right]$.
- **English Logic:** r: Not all even integers are odd.
- **Negation:** ~r: $\sim\sim \left[\forall x \left(f(x) \longrightarrow Q(x)\right)\right] = \left[\forall x \left(f(x) \longrightarrow Q(x)\right)\right]$.
- **Negation in words:** All even integer numbers are odd.

**Remark 1.6.15.**
Sometimes the English sentences are **unclear** with respect to quantification, or in another wards, quantified statements are often misused in **casual (informal) conversation**.

For example:
**(i) "If you can solve any problem we come up with, then you get an A for the course"**
The phrase **"you can solve any problem we can come up with"** could reasonably be interpreted as either a universal or existential quantification:

**(a) "you can solve every problem we come up with",**

**(b) "you can solve at least one problem we come up with".**

**(ii)** r**:** All students do not pay full tuition.
Here " r " could reasonably be interpreted as
**(a)** Not all students pay full tuition (Or: There exist some students do not pay full tuition).
**(b)** No students are pay full tuition (Or: There are no students are pay full tuition).
**Mathematical context:** Not all students pay full tuition.

**(iii)** r: All integer numbers are not even."
**(a)** Not all integer numbers are even.
**(b)** No integer numbers are even (Or: There are no even integers).
**Mathematical context:** Not all integer numbers are even.

**Combined Quantifiers 1.6.16**. There are six ways in which the quantifiers can be combined when two variables are present:

**(1)** $\forall\, x\, \forall\, y, f(x,y) \equiv\ \forall\, y\, \forall\, x, f(x,y)$=For every $x$, for every $y$, $f(x,y)$.

**(2)** $\forall\, x\, \exists\, y, f(x,y) \equiv$ For every $x$, there exists a $y$ such that $f(x,y)$.

**(3)** $\forall\, y\, \exists\, x, f(x,y) \equiv$ For every $y$, there exists an $x$ such that $f(x,y)$.

**(4)** $\exists\, x\, \forall\, y, f(x,y)$ $\equiv$There exists an $x$ such that for every $y$, $f(x,y)$.

**(5)** $\exists\, y\ \forall\, x\ f(x,y) \equiv$ There exists a $y$ such that for every $x$, $f(x,y)$.

**(6)** $\exists\, x\ \exists\, y, f(x,y) \equiv\ \exists\, y\ \exists\, x, f(x,y)$= There exists an $x$ such that there exists a $y$, $f(x,y)$.

**Example 1.6.17.**

**(i)** r: $\exists\, x \in \mathbb{R}\ \exists\, y \in \mathbb{R} : P(x,y) = (x^2 + y^2 = 2xy)$. The proposition " r " is true since $x = y = 1$ is one of many solutions.

**(ii)** s: $\forall\, x \in \mathbb{R}\ \ \exists\, y \in \mathbb{R} : P(x,y) = (y^3 = x)$. The proposition " s " is true since $y = \sqrt[3]{x}$ is solution for $P(x,y)$.

**(iii)** s: $\exists\, x \in \mathbb{R}\ \ \forall y \in \mathbb{R} : P(x,y) = (y^3 = x)$. Here "s" mean there is an "$x$" real such that for every "$y$" real, $P(x,y)$ is true. The proposition " s " is not true since no real numbers have this property.
**(iv)** r: For all $x$, there exists $y$ such that $xy = 1$.
**Solution.**

- $\forall \boldsymbol{x}, \boldsymbol{P}(\boldsymbol{x})$ **form:** r: $\forall\, x$, $\exists\, y$ such that $xy = 1$.
- **Negation:** ~r: ~ ($\forall\, x$, $\exists\, y$ such that $xy = 1$)
$\equiv\ \exists x$, ~ ($\exists\, y$ such that $(xy = 1)$)
$\equiv\ \exists x$, $\forall\, y$ such that $xy \neq 1$.
- **Negation in words:** ~r: There exists $x$ such that for all $y$, $xy \neq 1$.

**(v)** The following are equivalents.

**(a)** $\sim[\forall\, x\ \forall\, y, f(x,y)] \equiv \exists x\ \ \exists y, \sim f(x,y)$.

**(b)** $\sim[\exists\, x\ \ \exists y, f(x,y)] \equiv \forall x\ \ \forall y, \sim f(x,y)$.

**(c)** $\sim[\forall\, x\ \exists\, y, f(x,y)] \equiv \exists x\ \ \forall y, \sim f(x,y)$.

**(d)** $\sim[\exists\, x\ \forall\, y, f(x,y)] \equiv \forall x\ \ \exists y, \sim f(x,y)$.

**Solution. Exercise.**

## 1.7. Logical Reasoning

**Definition 1.7.1. (Arguments)**
An **argument** is a series of statements starting from hypothesis (premises/ assumptions) and ending with the conclusion.

From the definition, an argument might be valid or invalid.

**Definition 1.7.2. (Valid Arguments)(Proofs)**
An argument is said to be **valid** if the hypothesis implies the conclusion; that is, if $s$ is a statement implies from the statements $s_1, s_2, \dots, s_n$, then write as

$$s_1, s_2, \dots, s_n \mapsto s.$$

**Note 1.7.3.** In mathematics, the word ***proof*** is used to mean simply a valid argument. Many proofs involve more than two premises and a conclusion.

**Example 1.7.4.**
**(i)** Let $s_1$: Some mathematicians are engineering
$s_2$ : Ali is mathematician
$s$ : Ali is engineering
Show that the argument is valid.
**Solution.**
The argument $s_1, s_2 \mapsto s$ is not valid, since not all mathematicians are engineering.

**(ii)** Let $s_1$: There is no lazy student
$s_2$ : Ali is artist
$s_3$ : All artist are lazy
Find a conclusion $s$ for the above premises making the argument $s_1, s_2, s_3 \mapsto s$ is valid.
**Solution.**
Ali is----------------------.

**Remark 1.7.5.**
**(i)** An argument $\boxed{s_1, s_2, \dots, s_n \mapsto s}$
is valid if and only if $\boxed{(s_1 \wedge s_2 \wedge \dots \wedge s_n) \rightarrow s}$ is tautology; that is,

$$\boxed{(s_1 \wedge s_2 \wedge \dots \wedge s_n) \Longrightarrow s.}$$

**(ii)** An argument does not depend on the truth of the premises or the conclusion but it just interested only in the question

"**Is the conclusion implied by the conjunction of the premises?**"