Lecture 2

Rotating Coordinate Systems

2.1 Frame of Reference

In order to look at particle dynamics in the context of the atmosphere, we must deal with the fact that we live and observe the weather in a non-inertial reference frame.

2.2 Rotating and Non-rotating Frames of References

There are two coordinate systems when dealing with problems related to the earth:

a) One fixed to the earth that rotates and is thus accelerating (non-inertial), our real life frame of reference

b) One fixed with respect to the remote "star", i.e., an inertial frame where the Newton's laws are valid.

Inertia: a tendency to do nothing or to remain unchanged.

2.3 Atmospheric Forces

There are five atmospheric forces that drive the motion of the atmosphere:

1. Real forces: {The horizontal and vertical gradient pressure force, the gravitational force, the friction force},
2. Apparent forces: {Coriolis force, centrifugal force}.

This force does not arise from any physical interaction between two objects, but rather from the acceleration a of the non-inertial reference frame itself.

2.4 Total Derivative of a Vector in a Rotating System

For most applications in meteorology it is desirable to refer the motion to a reference frame rotating with the earth. Transformation of the momentum equation (Newton's second law of motion) to a rotating coordinate system requires a relationship between the total derivative of a vector in an inertial reference frame and the corresponding total derivative in a rotating system.

Note: The Newton laws do not applied in the non-inertial frame

To derive this relationship, we let $\vec{A}$ be an arbitrary vector whose Cartesian components in an <u>inertial</u> frame are given by:

$$\vec{A} = i'A'_x + j'A'_y + k'A'_z \quad (in\ inertial\ coordinates)$$

And whose components in a frame rotating with an angular velocity $\vec{\Omega}$ are:

$$\vec{A} = iA_x + jA_y + kA_z \quad (in\ non\ inertial\ coordinates) \qquad (2.1)$$

Letting $d_a\vec{A}/dt$ be the total derivative of $\vec{A}$ in the inertial frame, we can write

$$\frac{d_a\vec{A}}{dt} = i'\frac{dA'_x}{dt} + j'\frac{dA'_y}{dt} + k'\frac{dA'_z}{dt}$$

Note: The components are the same but the orientation is different

$$= i\frac{dA_x}{dt} + j\frac{dA_y}{dt} + k\frac{dA_z}{dt} + \frac{d_a i}{dt}A_x + \frac{d_a j}{dt}A_y + \frac{d_a k}{dt}A_z$$

where $\frac{d\vec{A}}{dt} \equiv i\frac{dA_x}{dt} + j\frac{dA_y}{dt} + k\frac{dA_z}{dt}$

which is just the total derivative of $\vec{A}$ as viewed in the rotating coordinates (i.e., the rate of change of $\vec{A}$ following the relative motion).

The last three terms arise because the directions of the unit vectors (i, j, k) change their orientation in space as the earth rotates.

For example, considering the eastward directed unit vector:

$$di = \frac{\partial i}{\partial \lambda}d\lambda + \frac{\partial i}{\partial \phi}d\phi + \frac{\partial i}{\partial z}dz$$

For solid body rotation, $d\lambda = \Omega dt,\ d\phi = 0,\ dz = 0$, so that:

$\frac{di}{dt} = \left(\frac{\partial i}{\partial \lambda}\right)\left(\frac{d\lambda}{dt}\right)$ then $\frac{d_a i}{dt} = \Omega\left(\frac{\partial i}{\partial \lambda}\right)$

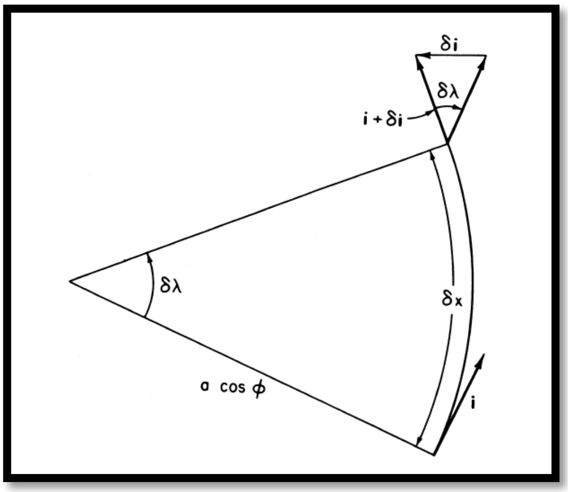

Fig. 2.1 Longitudinal dependence of **i**

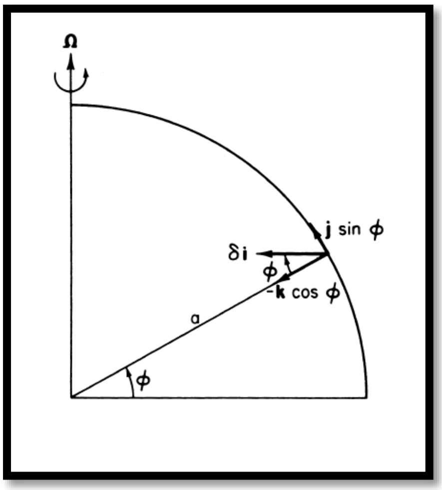

Fig. 2.2 Resolution of $\boldsymbol{\delta i}$ in Fig. 2.1 into northward and vertical components

From Figs. 2.1 and 2.2, the longitudinal derivative of i can be expressed as:

$$\frac{\partial i}{\partial \lambda} = j \; sin\, \phi - k \; cos\, \phi$$

However, $\vec{\Omega} = (0, \Omega \sin\phi\,, \Omega \cos\phi)$ so that:

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega} \times \vec{A} \qquad (2.2)$$

2.5 The Vectorial Form of the Momentum Equation in Rotating Coordinates

In an inertial reference frame, Newton's second law of motion may be written symbolically as:

$$\frac{d_a\vec{V}_a}{dt} = \sum \vec{F} \qquad (2.3)$$

The left-hand side represents the rate of change of the absolute velocity $\vec{V}_a$, following the motion as viewed in an inertial system. The right-hand side represents the sum of the real forces acting per unit mass. Now to find a relationship for the rotating system we apply Equation (2.2) to position vector $\vec{r}$ for an air parcel on the rotating earth:

$$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r} \qquad (2.4)$$

$$\vec{V}_a = \vec{V} + \vec{\Omega} \times \vec{r} \qquad (2.5)$$

which states simply that the absolute velocity of an object on the rotating earth is equal to its velocity relative to the earth plus the velocity due to the rotation of the earth.

Now we apply (Equation 2.2) to the velocity vector $\vec{V}_a$ and obtain:

$$\frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}_a}{dt} + \vec{\Omega} \times \vec{V}_a \qquad (2.6)$$

Substituting from (Equation 2.5) into the right-hand side of (Equation 2.6) gives

$$\frac{d_a\vec{V}_a}{dt} = \frac{d}{dt}\left(\vec{V} + \vec{\Omega} \times \vec{r}\,\right) + \vec{\Omega} \times \left(\vec{V} + \vec{\Omega} \times \vec{r}\,\right) \qquad (2.7a)$$

$$= \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \Omega^2\vec{R} \qquad (2.7b)$$

where $\vec{\Omega}$ is assumed to be constant. Here $\vec{R}$ is a vector perpendicular to the axis of rotation, with magnitude equal to the distance to the axis of rotation, so that with the aid of a vector identity,

$$\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\,\right) = \vec{\Omega} \times \left(\vec{\Omega} \times \vec{R}\,\right) = -\Omega^2\vec{R}$$

Equation (2.7) states that the acceleration following the motion in an inertial system equals the rate of change of relative velocity following the relative motion in the rotating frame plus the Coriolis acceleration due to relative motion in the rotating frame plus the centripetal acceleration caused by the rotation of the coordinates.

If we assume that the only real forces acting on the atmosphere are the pressure gradient force, gravitation, and friction, we can rewrite Newton's second law (2.3) with the aid of (2.7) as:

$$\frac{d_a\vec{V}_a}{dt} = \frac{1}{\rho}\nabla \mathrm{p} - \vec{\mathrm{g}} - \vec{\mathrm{F}}_r$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla \mathrm{p} + \vec{\mathrm{g}} + \vec{\mathrm{F}}_r \qquad (2.8)$$

where $\vec{\mathrm{F}}_r$ designates the frictional force, and <u>the centrifugal force has been combined with gravitation in the gravity term g</u>. Equation (2.8) is the statement of Newton's second law for motion relative to a rotating coordinate frame. It states that the acceleration following the relative motion in the rotating frame equals the sum of the Coriolis force, the pressure gradient force, effective gravity, and friction. This form of the momentum equation is basic to most work in dynamic meteorology.