Lecture 3

Derivation of the Momentum Equations of the Rotating Coordinates

3.1 Preface

In Lecture 2, we talked about the rotating coordinates system, which is fixed, to the earth. We found a relationship for the total derivative between the non-inertial (rotating) coordinates and the inertial (non-rotating) coordinates, which will be useful to derive the momentum equation in the atmospheric dynamics.

3.2 Full Derivation

Question: Derive the victorial equation of the momentum for a rotating system.

Sol. From the equation of the total derivative of a vector in a rotating system:

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega} \times \vec{A} \qquad (1)$$

and the Newton's 2nd law of motion in absolute reference:

$$\frac{d_a\vec{V}_a}{dt} = \sum \vec{F} \qquad (2)$$

Apply equ. 1 to the position vector ($\vec{r}$):

$$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r} \qquad (3)$$

$$\therefore \quad \vec{V}_a = \vec{V} + \vec{\Omega} \times \vec{r} \qquad (4)$$

$\vec{\boldsymbol{V}}$ is the velocity relative the coordinates, $\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{r}}$ the velocity of the coordinate system itself

Now, we apply equ. 1 to the velocity vector ($\vec{V}_a$):

$$\frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}_a}{dt} + \vec{\Omega} \times \vec{V}_a \qquad (5)$$

Substitute equ. 4 in equ. 5:

$$\frac{d_a\vec{V}_a}{dt} = \frac{d}{dt}\left[\vec{V} + \vec{\Omega}\times\vec{r}\right] + \vec{\Omega}\times[\vec{V} + \vec{\Omega}\times\vec{r}] \qquad (6)$$

$$\frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + \frac{d}{dt}(\vec{\Omega}\times\vec{r}) + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \qquad (7)$$

$$\frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + \left(\frac{d\vec{\Omega}}{dt}\times\vec{r}\right) + \left(\vec{\Omega}\times\frac{d\vec{r}}{dt}\right) + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \qquad (8)$$

$$\left(\frac{d\vec{\Omega}}{dt}\times\vec{r}\right) = 0 \quad \text{"}a\ derivative\ of\ a\ constant\text{"}$$

$$\therefore \quad \frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + \left(\vec{\Omega}\times\frac{d\vec{r}}{dt}\right) + \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \qquad (9)$$

$$Thus \quad \frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + 2(\vec{\Omega}\times\vec{V}) + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \qquad (9)$$

By using the vector triple product $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ to the last term in equ. 9 we get:

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = (\vec{\Omega}\cdot\vec{r})\vec{\Omega} - (\vec{\Omega}\cdot\vec{\Omega})\vec{r} = -\Omega^2\vec{R} \qquad (10)$$

$$Thus \quad \sum\vec{F} = \frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + 2(\vec{\Omega}\times\vec{V}) - \Omega^2\vec{R} \qquad (11)$$

If the only forces acting on the atmosphere are:

(i) pgf (ii) gravitation (iii)friction

we rewrite Newton's second law with the aid of equ. 11:

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r \qquad (12)$$

Where, the first term on the right is the Coriolis force, the second term on the right is the pressure gradient force (pgf), the third term on the right is the effective gravity (gravitation force + centrifugal force), the fourth term on right is the frictional force. **This form of the momentum equation is the basic to most work in dynamic meteorology.**