Lecture 4

Rotating Coordinate Systems

4.1 The Viscous Force

- The viscous force is due to the friction caused by interactions of molecules of the fluid. If we consider a layer of incompressible fluid between two horizontal plates separated by a distance (L) as shown in figure (4.1).

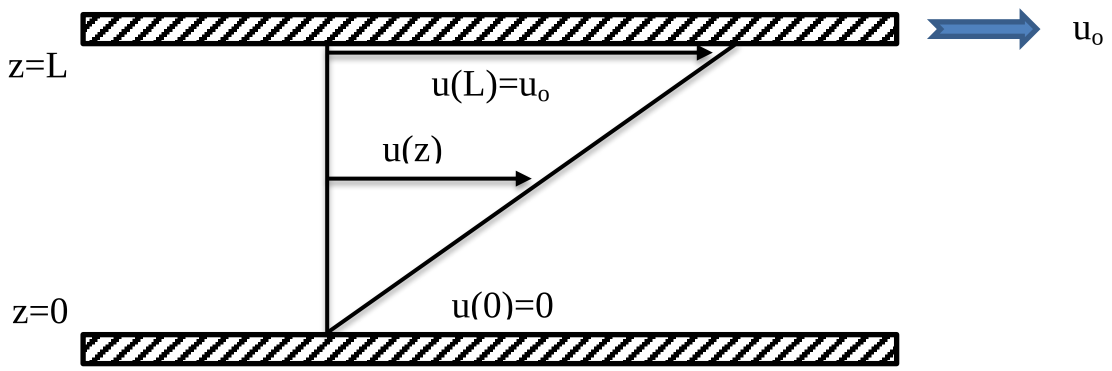

Figure (4.1)

- The layer in contact with the upper and lower plates will moved as the following:
  - ✓ At z=L the fluid moves at speed $u(L)=u_o$ in x-direction

    At z=0 the fluid is motionless $u(0)=0$.
  - ✓ The force exerted on the upper plate is:

$$F \propto \frac{Au_o}{L}$$

where A is area of the plate, L is the distance (depth) between the plates.

$$F = \mu \frac{Au_o}{L}$$

where μ is the dynamic viscosity coefficient.

- For a state of uniform motion, every horizontal layer of fluid of depth δz must exert the same force F on the fluid below. This may be expressed in the form:

$$F = \mu A \frac{\partial u}{\partial z}$$

$$\frac{\delta u}{\delta z} \equiv \frac{\Delta u}{\Delta z}$$

$$\tau_{zx} = \lim_{\delta z \to 0} \mu \frac{\delta u}{\delta z} = \mu \frac{\partial u}{\partial z}$$

where subscripts indicate that τ_{zx} is the component of the shearing stress in the x-direction due to vertical shear of the x velocity component.

- ✓ The downward momentum transport per unit time per unit area is simply the *shearing stress*.
- ✓ In our example there is no net viscous force acting on the elements of fluid, as the shearing stress acting across the top boundary of each fluid element is just equal and opposite to that acting across the lower boundary.

4.2 Calculating the Viscous Force

- Consider a differential volume element (parcel) of fluid centered at (x, y, z) with sides δxδyδz as shown in Figure (4.2). If the shearing stress in the x direction acting through the center of the element is designated τ_{zx}, then the stress acting across the upper boundary on the fluid below may be written approximately as

$$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \frac{\delta z}{2}$$

while the stress acting across the lower boundary on the fluid above is:

$$-\left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z} \frac{\delta z}{2}\right]$$

(This is just equal and opposite to the stress acting across the lower boundary on the fluid below.)

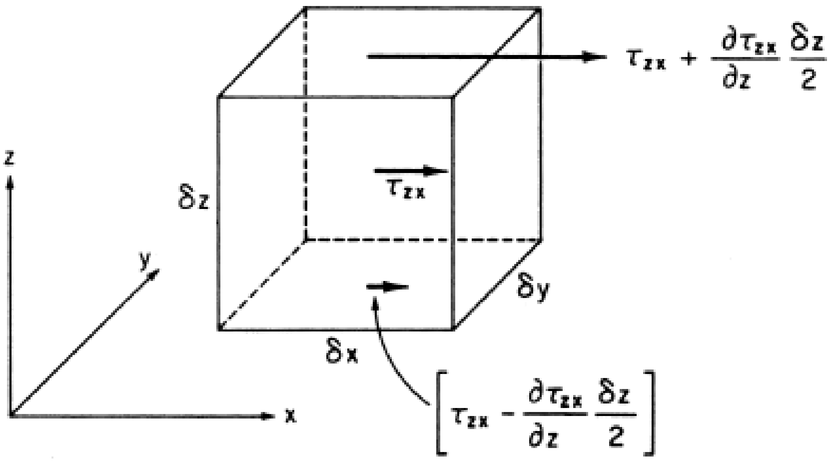

Figure (4.2)

The **net viscous force** on the volume element acting in the **x direction** is then given by the sum of the stresses acting across the upper boundary on the fluid below and across the lower boundary on the fluid above:

$$F_{zx} = A\sum \tau_{zx}$$

$$\left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]\delta x\delta y - \left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]\delta x\delta y$$

$$F_{zx} = \frac{\partial \tau_{zx}}{\partial z}\delta x\delta y\delta z$$

Dividing by mass $\rho\delta x\delta y\delta z$,

$$f_{zx} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \qquad (4.1)$$

$$Also \qquad f_{zx} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \equiv \nu\frac{\partial^2 u}{\partial z^2} \qquad (4.2)$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient

For standard atmospheric conditions $\nu = 1.46 \times 10^{-5} m^2 s^{-1}$

Now for y-direction $f_{zy} = \frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial z}$ and for z-direction $f_{zz} = \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}$

The total force per unit mass:

$$\vec{f} = f_{zx} + f_{zy} + f_{zz}$$

$$\vec{f} = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}$$

$$\vec{f} = \frac{1}{\rho}\frac{d\vec{\tau}}{dz}$$

Homework:

1. Why the fluid is considered incompressible in most atmospheric dynamics? When can we take into account the hypothesis that the fluid is compressible?
2. Derive the viscous force.