Chapter (2) Algebra of Vectors

Vectors can be added together and multiplied by scalars. Vector addition is associative (Equation 1) and commutative (Equation 2), and vector multiplication by a sum of scalars is distributive (Equation 3).

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}, \qquad (1)$$

and **associative**,

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}}). \qquad (2)$$

Moreover, multiplication by a scalar is **distributive**:

$$\alpha_1 \vec{\mathbf{A}} + \alpha_2 \vec{\mathbf{A}} = (\alpha_1 + \alpha_2) \vec{\mathbf{A}}. \qquad (3)$$

In addition, scalar multiplication by a sum of vectors is distributive:

$$\alpha(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = \alpha \vec{\mathbf{A}} + \alpha \vec{\mathbf{B}}. \qquad (4)$$

In this equation, α is any number (a scalar). For example, a vector antiparallel to vector $\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$ can be expressed simply by multiplying $\vec{\mathbf{A}}$ by the scalar $\alpha = -1$:

$$-\vec{\mathbf{A}} = -A_x \hat{\mathbf{i}} - A_y \hat{\mathbf{j}} - A_z \hat{\mathbf{k}}. \qquad (5)$$

Example 1

Direction of Motion

In a Cartesian coordinate system where $\hat{\mathbf{i}}$ denotes geographic east, $\hat{\mathbf{j}}$ denotes geographic north, and $\hat{\mathbf{k}}$ denotes altitude above sea level, a military convoy advances its position through unknown territory with velocity $\vec{\mathbf{v}} = (4.0\,\hat{\mathbf{i}} + 3.0\,\hat{\mathbf{j}} + 0.1\,\hat{\mathbf{k}})\text{km/h}$. If the convoy had to retreat, in what geographic direction would it be moving?

Solution

The velocity vector has the third component $\vec{\mathbf{v}}_z = (+0.1\text{km/h})\hat{\mathbf{k}}$, which says the convoy is climbing at a rate of 100 m/h through mountainous terrain. At the same time, its velocity is 4.0 km/h to the east and 3.0 km/h to the north, so it moves on the ground in direction $\tan^{-1}(3/4) \approx 37°$ north of east. If the convoy had to retreat, its new velocity vector $\vec{\mathbf{u}}$ would have to be antiparallel to $\vec{\mathbf{v}}$ and be in the form $\vec{\mathbf{u}} = -\alpha\vec{\mathbf{v}}$, where α is a positive number. Thus, the velocity of the retreat would be $\vec{\mathbf{u}} = \alpha(-4.0\,\hat{\mathbf{i}} - 3.0\,\hat{\mathbf{j}} - 0.1\,\hat{\mathbf{k}})\text{km/h}$. The negative sign of the third component indicates the convoy would be descending. The direction angle of the retreat velocity is $\tan^{-1}(-3\alpha/-4\alpha) \approx 37°$ south of west. Therefore, the convoy would be moving on the ground in direction $37°$ south of west while descending on its way back.

The generalization of the number zero to vector algebra is called the **null vector**, denoted by $\vec{\mathbf{0}}$. All components of the null vector are zero, $\vec{\mathbf{0}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$, so the null vector has no length and no direction.

Two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are **equal vectors** if and only if their difference is the null vector:

$$\vec{\mathbf{0}} = \vec{\mathbf{A}} - \vec{\mathbf{B}} = (A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}) - (B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}) = (A_x - B_x)\,\hat{\mathbf{i}} + (A_y - B_y)\,\hat{\mathbf{j}} + (A_z - B_z)\,\hat{\mathbf{k}}.$$

This vector equation means we must have simultaneously $A_x - B_x = 0$, $A_y - B_y = 0$, and $A_z - B_z = 0$. Hence, we can write $\vec{\mathbf{A}} = \vec{\mathbf{B}}$ if and only if the corresponding components of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are equal:

$$\vec{\mathbf{A}} = \vec{\mathbf{B}} \Leftrightarrow \begin{cases} A_x = B_x \\ A_y = B_y. \\ A_z = B_z \end{cases} \tag{6}$$

Two vectors are equal when their corresponding scalar components are equal. Resolving vectors into their scalar components (i.e., finding their scalar components) and expressing them analytically in vector component form (given by Equation 7)

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}. \tag{7}$$

Allows us to use vector algebra to find sums or differences of many vectors analytically (i.e., without using graphical methods). For example, to find the resultant of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, we simply add them component by component, as follows:

$$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) + (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}.$$

In this way, using **Equation 6** scalar components of the resultant vector $\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$ are the sums of corresponding scalar components of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$:

$$\begin{cases} R_x = A_x + B_x, \\ R_y = A_y + B_y, \\ R_z = A_z + B_z. \end{cases}$$

Analytical methods can be used to find components of a resultant of many vectors. For example, if we are to sum up N vectors $\vec{\mathbf{F}}_1, \vec{\mathbf{F}}_2, \vec{\mathbf{F}}_3, \ldots, \vec{\mathbf{F}}_N$, where each vector is $\vec{\mathbf{F}}_k = F_{kx}\hat{\mathbf{i}} + F_{ky}\hat{\mathbf{j}} + F_{kz}\hat{\mathbf{k}}$, the resultant vector $\vec{\mathbf{F}}_R$ is

$$\begin{aligned} \vec{\mathbf{F}}_R = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \ldots + \vec{\mathbf{F}}_N &= \sum_{k=1}^{N} \vec{\mathbf{F}}_k = \sum_{k=1}^{N} \left(F_{kx}\hat{\mathbf{i}} + F_{ky}\hat{\mathbf{j}} + F_{kz}\hat{\mathbf{k}}\right) \\ &= \left(\sum_{k=1}^{N} F_{kx}\right)\hat{\mathbf{i}} + \left(\sum_{k=1}^{N} F_{ky}\right)\hat{\mathbf{j}} + \left(\sum_{k=1}^{N} F_{kz}\right)\hat{\mathbf{k}}. \end{aligned}$$

Therefore, scalar components of the resultant vector are

$$\begin{cases} F_{Rx} = \sum_{k=1}^{N} F_{kx} = F_{1x} + F_{2x} + \ldots + F_{Nx} \\ F_{Ry} = \sum_{k=1}^{N} F_{ky} = F_{1y} + F_{2y} + \ldots + F_{Ny} \\ F_{Rz} = \sum_{k=1}^{N} F_{kz} = F_{1z} + F_{2z} + \ldots + F_{Nz}. \end{cases} \qquad \boxed{8}$$

Having found the scalar components, we can write the resultant in vector component form:

$$\vec{\mathbf{F}}_R = F_{Rx}\hat{\mathbf{i}} + F_{Ry}\hat{\mathbf{j}} + F_{Rz}\hat{\mathbf{k}}.$$

Analytical methods for finding the resultant and, in general, for solving vector equations are very important in physics because many physical quantities are vectors. For example, we use this method in kinematics to find resultant displacement vectors and resultant velocity vectors, in mechanics to find resultant force vectors and the resultants of many derived vector quantities, and in electricity and magnetism to find resultant electric or magnetic vector fields.

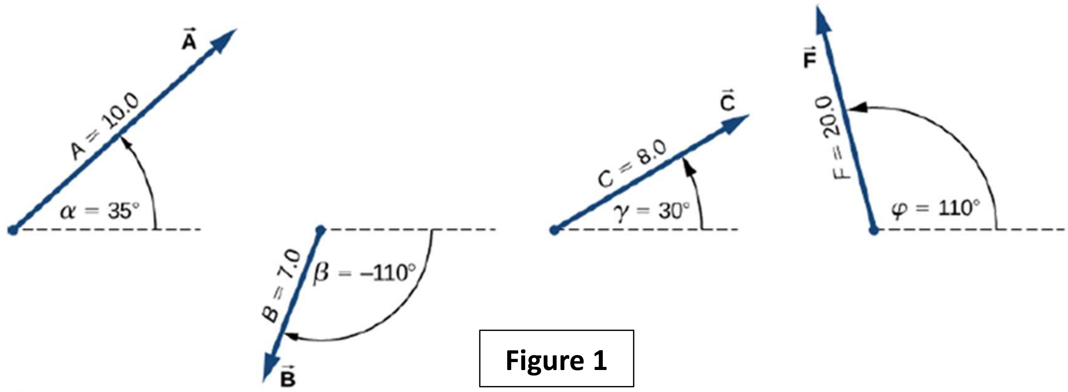


Figure 1

Example 2

Analytical Computation of a Resultant

Three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in a plane (**Figure 1**) are specified by their magnitudes $A = 10.0$, $B = 7.0$, and $C = 8.0$, respectively, and by their respective direction angles with the horizontal direction $\alpha = 35°$, $\beta = -110°$, and $\gamma = 30°$. The physical units of the magnitudes are centimeters. Resolve the vectors to their scalar components and find the following vector sums: (a) $\vec{R} = \vec{A} + \vec{B} + \vec{C}$, (b) $\vec{D} = \vec{A} - \vec{B}$, and (c) $\vec{S} = \vec{A} - 3\vec{B} + \vec{C}$.

Solution

We resolve the given vectors to their scalar components:

$$\begin{cases} A_x = A\cos\alpha = (10.0\text{ cm})\cos 35^\circ = 8.19\text{ cm} \\ A_y = A\sin\alpha = (10.0\text{ cm})\sin 35^\circ = 5.73\text{ cm} \end{cases}$$
$$\begin{cases} B_x = B\cos\beta = (7.0\text{ cm})\cos(-110^\circ) = -2.39\text{ cm} \\ B_y = B\sin\beta = (7.0\text{ cm})\sin(-110^\circ) = -6.58\text{ cm} \end{cases}.$$
$$\begin{cases} C_x = C\cos\gamma = (8.0\text{ cm})\cos 30^\circ = 6.93\text{ cm} \\ C_y = C\sin\gamma = (8.0\text{ cm})\sin 30^\circ = 4.00\text{ cm} \end{cases}$$

For (a) we may substitute directly into Equation 8 to find the scalar components of the resultant:

$$\begin{cases} R_x = A_x + B_x + C_x = 8.19\text{ cm} - 2.39\text{ cm} + 6.93\text{ cm} = 12.73\text{ cm} \\ R_y = A_y + B_y + C_y = 5.73\text{ cm} - 6.58\text{ cm} + 4.00\text{ cm} = 3.15\text{ cm} \end{cases}.$$

Therefore, the resultant vector is $\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} = (12.7\hat{\mathbf{i}} + 3.1\hat{\mathbf{j}})\text{cm}$.

For (b), we may want to write the vector difference as

$$\vec{\mathbf{D}} = \vec{\mathbf{A}} - \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}) - (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}) = (A_x - B_x)\hat{\mathbf{i}} + (A_y - B_y)\hat{\mathbf{j}}.$$

Then, the scalar components of the vector difference are

$$\begin{cases} D_x = A_x - B_x = 8.19\text{ cm} - (-2.39\text{ cm}) = 10.58\text{ cm} \\ D_y = A_y - B_y = 5.73\text{ cm} - (-6.58\text{ cm}) = 12.31\text{ cm} \end{cases}.$$

Hence, the difference vector is $\vec{\mathbf{D}} = D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}} = (10.6\hat{\mathbf{i}} + 12.3\hat{\mathbf{j}})\text{cm}$.

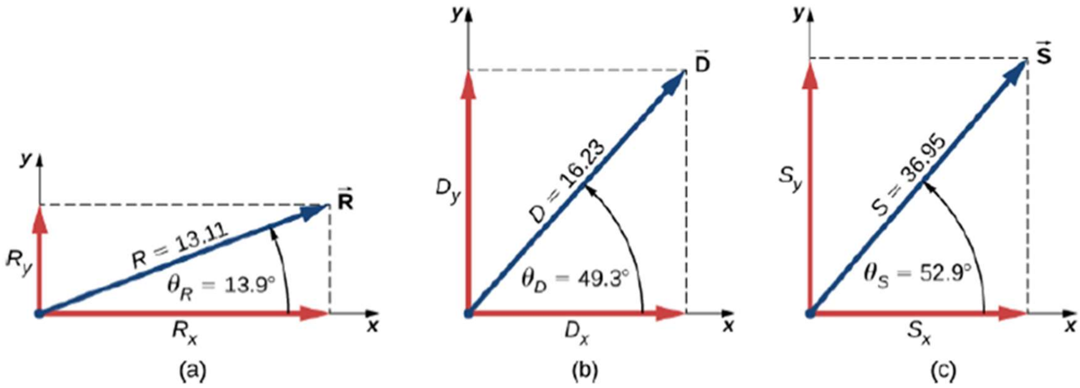
For (c), we can write vector $\vec{\mathbf{S}}$ in the following explicit form:

$$\begin{aligned} \vec{\mathbf{S}} &= \vec{\mathbf{A}} - 3\vec{\mathbf{B}} + \vec{\mathbf{C}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}) - 3(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}) + (C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}}) \\ &= (A_x - 3B_x + C_x)\hat{\mathbf{i}} + (A_y - 3B_y + C_y)\hat{\mathbf{j}}. \end{aligned}$$

Then, the scalar components of $\vec{\mathbf{S}}$ are

$$\begin{cases} S_x = A_x - 3B_x + C_x = 8.19\text{ cm} - 3(-2.39\text{ cm}) + 6.93\text{ cm} = 22.29\text{ cm} \\ S_y = A_y - 3B_y + C_y = 5.73\text{ cm} - 3(-6.58\text{ cm}) + 4.00\text{ cm} = 29.47\text{ cm} \end{cases}.$$

The vector is $\vec{\mathbf{S}} = S_x\hat{\mathbf{i}} + S_y\hat{\mathbf{j}} = (22.3\hat{\mathbf{i}} + 29.5\hat{\mathbf{j}})\text{cm}$.



$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \quad (9)$$

Example 3

Vector Algebra

Find the magnitude of the vector $\vec{C}$ that satisfies the equation $2\vec{A} - 6\vec{B} + 3\vec{C} = 2\hat{j}$, where $\vec{A} = \hat{i} - 2\hat{k}$ and $\vec{B} = -\hat{j} + \hat{k}/2$.

Strategy

We first solve the given equation for the unknown vector $\vec{C}$. Then we substitute $\vec{A}$ and $\vec{B}$; group the terms along each of the three directions $\hat{i}$, $\hat{j}$, and $\hat{k}$; and identify the scalar components C_x, C_y, and C_z. Finally, we substitute into Equation 9 to find magnitude C.

Solution

$$\begin{aligned}
2\vec{A} - 6\vec{B} + 3\vec{C} &= 2\hat{j} \\
3\vec{C} &= 2\hat{j} - 2\vec{A} + 6\vec{B} \\
\vec{C} &= \tfrac{2}{3}\hat{j} - \tfrac{2}{3}\vec{A} + 2\vec{B} \\
&= \tfrac{2}{3}\hat{j} - \tfrac{2}{3}(\hat{i} - 2\hat{k}) + 2\left(-\hat{j} + \frac{\hat{k}}{2}\right) = \tfrac{2}{3}\hat{j} - \tfrac{2}{3}\hat{i} + \tfrac{4}{3}\hat{k} - 2\hat{j} + \hat{k} \\
&= -\tfrac{2}{3}\hat{i} + \left(\tfrac{2}{3} - 2\right)\hat{j} + \left(\tfrac{4}{3} + 1\right)\hat{k} \\
&= -\tfrac{2}{3}\hat{i} - \tfrac{4}{3}\hat{j} + \tfrac{7}{3}\hat{k}.
\end{aligned}$$

The components are $C_x = -2/3$, $C_y = -4/3$, and $C_z = 7/3$, and substituting into Equation 9 gives

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2} = \sqrt{(-2/3)^2 + (-4/3)^2 + (7/3)^2} = \sqrt{23/3}.$$

Definition of the scalar product	$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\varphi$
Commutative property of the scalar product	$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$
Distributive property of the scalar product	$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{C}}$
Scalar product in terms of scalar components of vectors	$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$
Cosine of the angle between two vectors	$\cos\varphi = \dfrac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{AB}$
Dot products of unit vectors	$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$
Magnitude of the vector product (definition)	$\lvert \vec{\mathbf{A}} \times \vec{\mathbf{B}} \rvert = AB\sin\varphi$
Anticommutative property of the vector product	$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$
Distributive property of the vector product	$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$
Cross products of unit vectors	$\begin{cases} \hat{\mathbf{i}} \times \hat{\mathbf{j}} = +\hat{\mathbf{k}}, \\ \hat{\mathbf{j}} \times \hat{\mathbf{k}} = +\hat{\mathbf{i}}, \\ \hat{\mathbf{k}} \times \hat{\mathbf{i}} = +\hat{\mathbf{j}}. \end{cases}$
The cross product in terms of scalar components of vectors	$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$