Foundation of Mathematics I

Chapter 4   Functions

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Chapter Four

Functions

Definition 4.1. A **function** or a **mapping** from A to B, denoted by $f: A \to B$ is a relation f from A to B in which every element from A appears exactly once as the first component of an ordered pair in the relation. That is, each $a \in A$ the relation f contains exactly one ordered pair of form (a, b).

Equivalent statements to the function definition.

(i) A relation f from A to B is function iff

$$\forall x \in A\ \exists!\, y \in B \text{ such that } (x, y) \in f$$

(ii) A relation f from A to B is function iff

$$\forall x \in A\ \forall y, z \in B, \text{ if } (x, y) \in f \wedge (x, z) \in f, \text{ then } y = z.$$

(iii) A relation f from A to B is function iff

$$(x_1, y_1) \text{ and } (x_2, y_2) \in f \text{ such that if } x_1 = x_2, \text{ then } y_1 = y_2.$$

This property called **the well-defined relation**.

Example 4.2.

(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$.

(1) $R_1 = \{(1, 2), (2, 4), (3, 4), (4, 5)\}$ function from A to B.

(2) $R_2 = \{(1, 2), (2, 4), (2, 5), (4, 5)\}$ not a function.

(3) $R_3 = \{(1, 2), (2, 4), (4, 5)\}$ function from $\{1, 2, 4\}$ to B.

(4) $R_4 = A \times B$ not a function.

(ii) Consider the relations described below.

Relation	Orderd pairs	Sample Relation
1	(person, month)	$\{(A, May), (B, Dec), (C, Oct), ..\}$
2	(hours, pay)	$\{(12, 84), (4, 28), (6, 42), (15, 105), \dots\}$
3	(instructor, course)	$\{(A, MATH001), (A, MATH002), \dots\}$
4	(time, temperature)	$\{(8, 70^\circ), (10, 78^\circ), (12, 78^\circ), \dots\}$

The first relation is a function because each person has only one birth month.
The second relation is a function because the number of hours worked at a particular job can yield only one paycheck amount.
The third relation is not a function because an instructor can teach more than one course.
The fourth relation is a function. Note that the ordered pairs $(10, 78^\circ), (12, 78^\circ)$ do not violate the definition of a function.

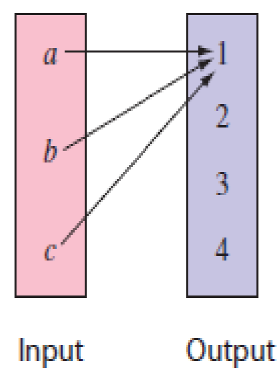
(iii) Decide whether each relation represents a function.

a. Input: a, b, c

Output: 2, 3, 4

$\{(a, 2), (b, 3), (c, 4)\}$

b.



c.

Input x	Output y	(x, y)
3	1	$(3, 1)$
4	3	$(4, 3)$
5	4	$(5, 4)$
3	2	$(3, 2)$

Solution.
a. This set of ordered pairs does represent a function. No first component has two different second components.
b. This diagram does represent a function. No first component has two different second components.
c. This table does not represent a function. The first component 3 is paired with two different second components, 1 and 2.

Notation 4.3. We write $f\,(a) = b$ when $(a, b) \in f$ where f is a function. We say that b is the **image** of a under f, and a is a **preimage** of b.

Definition 4.4. Let $f : A \to B$ be a function from A to B.

(i) The set A is called the **domain** of f, ($\boldsymbol{D(f)}$), and the set B is called the **codomain** of f, $\boldsymbol{C(f)}$.

(ii) The set $f\,(A) = \{f\,(x) \mid x \in A\}$ is called the **range** of f, ($\boldsymbol{R(f)}$).

Remark 4.5.

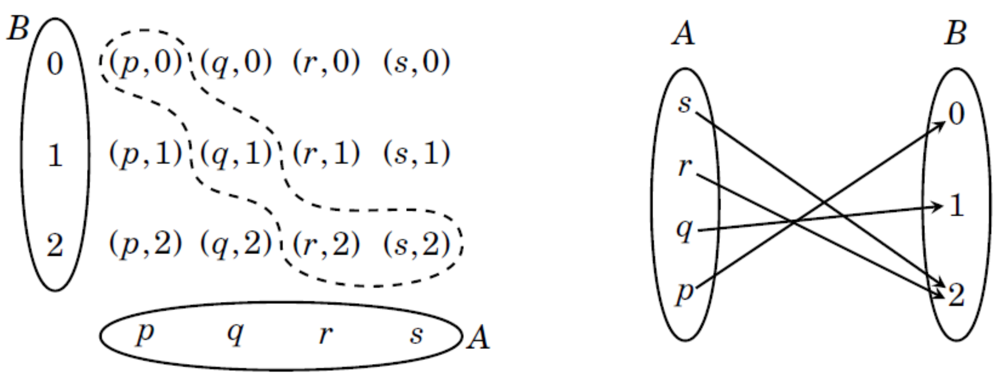
(i) Think of the domain as the set of possible **"input values"** for f.

(ii) Think of the range as the set of all possible **"output values"** for f.

Example 4.6.
(i) Let $A = \{p, q, r, s\}$ and $B = \{0,1,2\}$ and

$$f = \{(p,0),(q,1),(r,2),(s,2)\} \subseteq A \times B.$$

This is a function $f: A \to B$ because each element of A occurs exactly once as a first coordinate of an ordered pair in f.
We have $f\,(p) = 0, f\,(q) = 1, f\,(r) = 2$ and $f\,(s) = 2$. The domain of f is A, and the codomain and range are both B.



(a) (b)

Figure. Two ways of drawing the function $f = \{(p,0),(q,1),(r,2),(s,2)\}$

(ii) Say a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m,n) = 6m - 9n$.
Note that as a set, this function is

$$f = \{((m,n), 6m - 9n) : (m,n) \in \mathbb{Z} \times \mathbb{Z}\} \subseteq (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}.$$

What is the range of ?
To answer this, first observe that for any $(m,n) \in \mathbb{Z} \times \mathbb{Z}$, the value

$$f(m,n) = 6m - 9n = 3(2m - 3n)$$

is a multiple of 3. Thus every number in the range is a multiple of 3, so

$$R(f) \subseteq \{3k : k \in \mathbb{Z}\}. \qquad \cdots (1)$$

On the other hand if $b = 3k$ is a multiple of 3 we have

$$f(-k,-k) = 6(-k) - 9(-k) = -6k + 9k = 3k,$$

which means any multiple of 3 is in the range of f, so

$$\{3k: k \in \mathbb{Z}\} \subseteq R(f). \qquad \cdots (2)$$

Therefore, from (1) and (2) we get

$$R(f) = \{3k: k \in \mathbb{Z}\}.$$

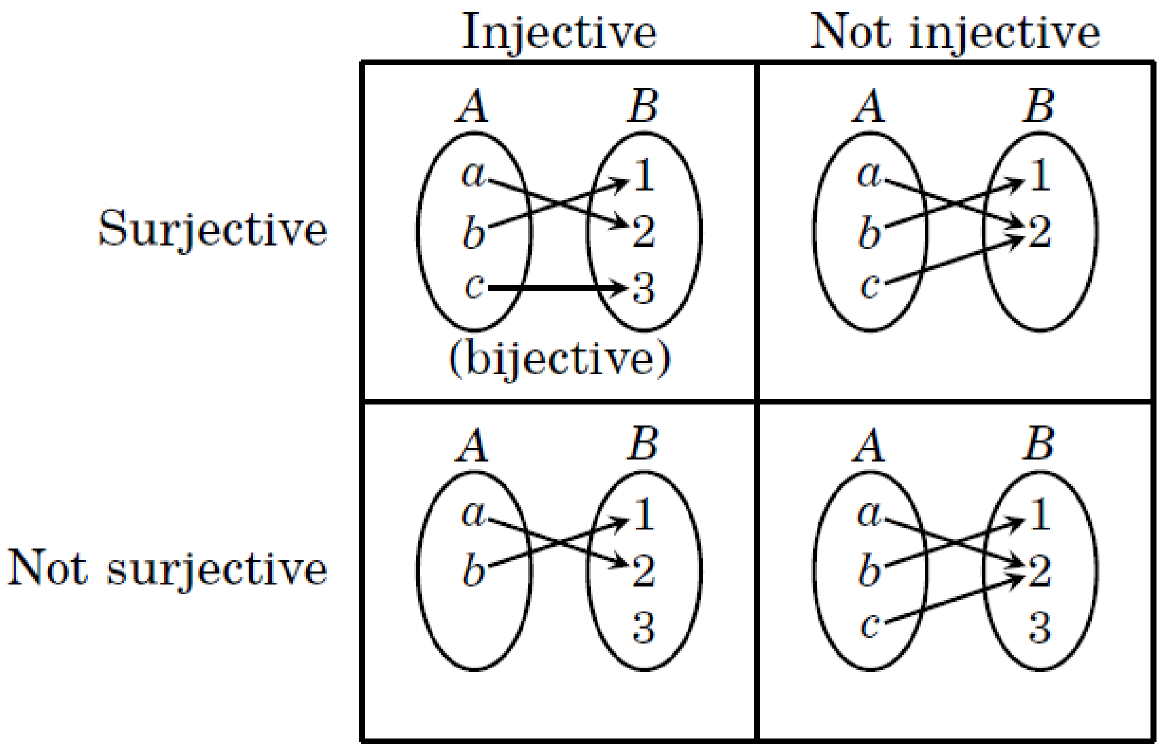
Definition 4.7. Two functions $f: A \to B$ and $g: C \to D$ are **equal** if $A = C, B = D$ and $f(x) = g(x)$ for every $x \in A$.

Example 4.8.
(i) Suppose that $A = \{1,2,3\}$ and $B = \{a, b\}$. The two functions $f = \{(1, a), (2, a), (3, b)\}$ and $g = \{(3, b), (2, a), (1, a)\}$ from A to B are equal because the sets f and g are equal. Observe that the equality $f = g$ means $f(x) = g(x)$ for every $x \in A$.
(ii) Let $f(x) = (x^2 - 1)/(x - 1)$ and $g(x) = x + 1$, where $x \in \mathbb{R}$.
$f(x) = (x - 1)(x + 1)/(x - 1) = (x + 1)$.
$D(f) = \mathbb{R} - \{1\}$, $R(f) = \mathbb{R} - \{2\}$.
$D(g) = \mathbb{R}$, $R(f) = \mathbb{R}$.
$f \neq g$.

Definition 4.9.
(i) A function $f: A \to B$ is **one-to-one** or **injective** if each element of B appears at most once as the image of an element of A. That is, a function $f: A \to B$ is injective if $\forall x, y \in A,\ f(x) = f(y) \Rightarrow x = y$ **or** $\forall x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$.

(ii) A function $f: A \to B$ is **onto** or **surjective** if $f(A) = B$, that is, each element of B appears at least once as the image of an element of A. That is, a function $f: A \to B$ is surjective if $\forall y \in B\ \exists x \in A$ such that $f(x) = y$.

(iii) A function $f : A \to B$ is **bijective** iff it is one-to-one and onto.



Example 4.10. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(x) = 3x + 7$.

$$f = \{\dots, (-3,-2), (-2,1), (-1,4), (0,7), (1,10), (2,13), \dots\}.$$

(i) f is injective. Suppose otherwise; that is,

$$f(x) = f(y) \Rightarrow 3x + 7 = 3y + 7 \Rightarrow 3x = 3y \Rightarrow x = y$$

(ii) f is not surjective. For $b = 2$ there is no a such that $f(a) = b$; that is, $2 = 3a + 7$ holds for $a = -\frac{5}{3}$ which is not in $\mathbb{Z} = D(f)$.

Example 4.11.
(i) Show that the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as
$f(x) = (1/x) + 1$ is injective but not surjective.
Solution.
We will use the contrapositive approach to show that f is injective.
Suppose $x, y \in \mathbb{R} - \{0\}$ and $f(x) = f(y)$. This means
$\frac{1}{x} + 1 = \frac{1}{y} + 1 \longrightarrow x = y$. Therefore f is injective.
Function f is not surjective because there exists an element $b = 1 \in \mathbb{R}$ for which
$f(x) = (1/x) + 1 \neq 1$ for every $x \in \mathbb{R}$.

(ii) Show that the function $f: \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by the formula $f(m,n) = (m+n, m+2n)$, is both injective and surjective.

Solution.

Injective: Let $(m,n),(r,s) \in \mathbb{Z} \times \mathbb{Z} = D(f)$ such that $f(m,n) = f(r,s)$. To prove $(m,n) = (r,s)$.

1-$f(m,n) = f(r,s) \Longrightarrow (m+n, m+2n) = (r+s, r+2s)$ Hypothesis
2- $m+n = r+s$ Def. of $\times$
3- $m+2n = r+2s$ Def. of $\times$
4- $m = r+2s-2n$ Inf. (3)
5- $n = s$ and $m = r$ Inf. (2),(4)
6- $(m,n) = (r,s)$ Def. of $\times$

Surjective: Let $(x,y) = \mathbb{Z} \times \mathbb{Z} = R(f)$. To prove $\exists (m,n) \in \mathbb{Z} \times \mathbb{Z} = D(f) \ni f(m,n) = (x,y)$.

1-$f(m,n) = (m+n, m+2n) = (x,y)$ Def. of f
2- $m+n = x$ Def. of $\times$
3- $m+2n = y$ Def. of $\times$
4- $m = x-n$ Inf. (2)
5- $n = y-x$ Inf. (3),(4)
6- $m = -x$ Inf. (2),(5)
7- $(-x, y-x) \in \mathbb{Z} \times \mathbb{Z} = D(f)$, $f(-x, y-x) = (x,y)$

Definition 4.12. The **composition** of functions $f: X \to Y$ with $g: Y \to Z$ is the function $g \circ f: X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.

Remark 4.13.

(i) The composition $g \circ f$ can only be defined if the domain of g includes the range of f; that is, $R(f) \subseteq D(g)$, and the existence of $g \circ f$ does not imply that $f \circ g$ even makes sense.

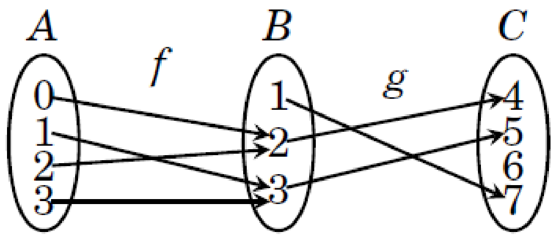
(ii) The order of application of the functions in a composition is crucial and is read from from right to left.

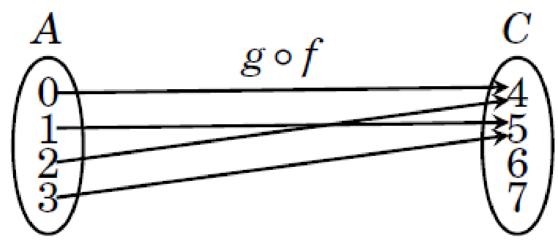
Example 4.14.

(i) Let $A = \{0,1,2,3\}$, $B = \{1,2,3\}$, $C = \{4,5,6,7\}$. If $f : A \to B$ and $g : B \to C$ are the functions defined as follows.

$f = \{(0,2),(1,3),(2,2),(3,3)\}$, $g = \{(1,7),(2,4),(3,5)\}$.

$g \circ f = \{(0,4), (1,5), (2,4), (3,5)\}$





(ii) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions defined as follows.

$f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $(g \circ f)(x) = g\left(f(x)\right) = g(x^2) = \sqrt{x^2}$.

Here $R(f) = [0, \infty) \subseteq D(g) = \mathbb{R}$.

Theorem 4.15.

(i) Suppose $f : A \to B$ and $g : B \to C$ be functions. If both f and g are injective, then $g \circ f$ is injective. If both f and g are surjective, then $g \circ f$ is surjective.

(ii) Composition of functions is associative. That is, if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Proof.

(i) To prove $g \circ f$ is 1-1. Let $x, y \in A$ and $(g \circ f)(x) = (g \circ f)(y)$.
To prove $x = y$.

$(g \circ f)(x) = g\left(f(x)\right) = g(f(y))$ Def. of $\circ$
$f(x) = f(y)$ Since g is 1-1 and Def. of 1-1on g
$x = y$ Since f is 1-1 and Def. of 1-1on f
$\therefore g \circ f$ is 1-1.

To prove $g \circ f$ is onto. Let $z \in D$, to prove $\exists\, x \in A$ such that $(g \circ f)(x) = z$.
(1) $\exists y \in B$ such that $g(y) = z$ Since g is onto and Def. of onto on g
(2) $\exists x \in A$ such that $f(x) = y$ Since f is onto and Def. of onto on f
$g\left(f(x)\right) = z$ Inf. (1), (2)
$(g \circ f)(x) = z$ Def. of $\circ$
$\therefore g \circ f$ is onto.

(ii) Exercise.

Theorem 4.16. Let $f : X \to Y$ be a function. Then f is bijective iff the inverse relation f^{-1} is a function from B to A.

Proof.

Suppose $f : X \to Y$ is bijective. To prove f^{-1} is a function from B to A.

$(*)$ Let (y_1, x_1) and $(y_2, x_2) \in f^{-1}$ such that $y_1 = y_2$, to prove $x_1 = x_2$.

(x_1, y_1) and $(x_2, y_2) \in f$	Def. of f^{-1}
(x_1, y_1) and $(x_2, y_1) \in f$	By hypothesis $(*)$
$x_1 = x_2$	Def. of 1-1 on f

$\therefore f^{-1}$ is a function from B to A.

Conversely, suppose f^{-1} is a function from B to A, to prove $f : X \to Y$ is bijective; that is, 1-1 and onto.

1-1: Let $a, b \in X$ and $f(a) = f(b)$. To prove $a = b$.

$(a, f(a))$ and $(b, f(b)) \in f$	Hypothesis (f is function)
$(a, f(a))$ and $(b, f(a)) \in f$	Hypothesis ($f(a) = f(b)$)
$(f(a), a)$ and $(f(a), b) \in f^{-1}$	Def. of inverse relation f^{-1}
$a = b$	Since f^{-1} is function

$\therefore f$ is 1-1.

onto: Let $b \in Y$. To prove $\exists a \in A$ such that $f(a) = b$.

$(b, f^{-1}(b)) \in f^{-1}$	Hypothesis (f^{-1} is a function from B to A)
$(f^{-1}(b), b) \in f$	Def. of inverse relation f^{-1}

Put $a = f^{-1}(b)$.

$a \in A$ and $f(a) = b$	Hypothesis (f is function)

$\therefore f$ is onto.

Definition 4.17.

(i) A function $I_A : A \rightarrow A$ defined by $I_A(x) = x$, for every $x \in A$ is called the **identity** function on A. $I_A = \{(x, x) : x \in A\}$.

(ii) Let $A \subseteq X$. A function $i_A : A \rightarrow X$ defined by $i_A(x) = x$, for every $x \in A$ is called the **inclusion** function on A.