2. Divided Differences Interpolation:

****Newton's Divided Difference Interpolating Polynomial:**

- The ***zeroth divided difference*** of the function f with respect to x_i, denoted $f[x_i]$, is simply the value of f at x_i :
$$f[x_i] = f(x_i)$$

- The ***first divided difference*** of f with respect to x_i and $x_{i+1}$is denoted $f[x_i, x_{i+1}]$ and defined as
$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

- The ***second divided difference***, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as
$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

- Have been determined, the ***kth divided difference*** relative to x_i, $x_{i+1}, x_{i+2}, \dots, x_{i+k}$ is
$$f[x_i, x_{i+1}, \dots, x_{i+k-1}, x_{i+k}] = \frac{f[x_i, x_{i+1}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

- The process ends with the single ***nth divided difference***,
$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0,x_1]=\dfrac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2]=\dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1,x_2]=\dfrac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3]=\dfrac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1,x_2,x_3]=\dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
		$f[x_2,x_3]=\dfrac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4]=\dfrac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$f[x_2,x_3,x_4]=\dfrac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3,x_4]=\dfrac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2,x_3,x_4,x_5]=\dfrac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
x_4	$f[x_4]$		$f[x_3,x_4,x_5]=\dfrac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4,x_5]=\dfrac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

Divided Difference Table

Hence, the interpolating polynomial is

$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots + f[x_0,x_1,\ldots,x_n](x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

So $P_n(x)$ can be rewritten in a form called ***Newton's Divided-Difference*** **or** ***(corresponding Collocation Polynomial):***

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0,x_1,\ldots,x_k](x-x_0)(x-x_1)\ldots(x-x_{k-1})$$

Example 1:

Consider the data in the following table:

x_i	0	1	3	6	10
$f(x_i)$	1	-6	4	169	921

Newton's Divided Difference Formula

First, we construct the divided-difference table from this data.

The divided differences in the table are computed as follows:

The ***zeroth divided difference***

$$f[x_0] = f(x_0) = 1$$
$$f[x_1] = f(x_1) = -6$$
$$f[x_2] = f(x_2) = 4$$
$$f[x_3] = f(x_3) = 169$$
$$f[x_4] = f(x_4) = 921$$

The ***first divided difference***

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{-6 - 1}{1 - 0} = -7$$
$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{4 - (-6)}{3 - 1} = 5$$
$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{169 - 4}{6 - 3} = 55$$
$$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3} = \frac{921 - 169}{10 - 6} = 188$$

The ***second divided difference***

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{5 - (-7)}{3 - 0} = 4$$
$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{55 - 5}{6 - 1} = 10$$
$$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} = \frac{188 - 55}{10 - 3} = 19$$

The ***third divided difference***

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{10 - 4}{6 - 0} = 1$$
$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1} = \frac{19 - 10}{10 - 1} = 1$$

The ***forth divided difference***

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0}$$
$$= \frac{1 - 1}{10 - 0} = 0$$

Second, The resulting divided-difference table is

x	$f(x)$	$1^{st}DD$	$2^{nd}DD$	$3^{rd}DD$	$4^{th}DD$
0	1				
		-7			
1	-6		4		
		5		1	
3	4		10		0
		55		1	
6	169		19		
		188			
10	921				

Third, It follows that the interpolating polynomial $P_n(x)$can be obtained using the Newton divided-difference formula as follows:

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$P_n(x) = 1 + (-7)(x - 0) + (4)(x - 0)(x - 1) + (1)(x - 0)(x - 1)(x - 3) + (0)(x - 0)(x - 1)(x - 3)(x - 6)$$

$$P_n(x) = 1 - 7x + 4x^2 - 4x + x^3 - 4x^2 + 3x$$

$$\therefore \; P_3(x) = x^3 - 8x + 1$$

To find $f(4)$

$$\because \; f(x) \approx P_3(x) = x^3 - 8x + 1$$

$$\therefore f(4) \approx (4)^3 - 8 * (4) + 1 = 33$$

Example 2:

Use Newton's divided difference formula to show that an interpolation for $\sqrt[3]{20}$ from the points

$(0,0), (1,1), (8,2), (27,3), (64,4)$, on $f(x) = \sqrt[3]{x}$ is quite invalid.

Solution:

$$f[x_0] = f(x_0) = 0$$
$$f[x_1] = f(x_1) = 1$$
$$f[x_2] = f(x_2) = 2$$
$$f[x_3] = f(x_3) = 3$$
$$f[x_4] = f(x_4) = 4$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{1 - 0}{1 - 0} = 1$$
$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{2 - 1}{8 - 1} = 0.142857$$
$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{3 - 2}{27 - 8} = 0.052632$$
$$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3} = \frac{4 - 3}{64 - 27} = 0.027027$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{0.142857 - 1}{8 - 0}$$
$$= -0.107143$$
$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{0.052632 - 0.142857}{27 - 1}$$
$$= -0.00347$$
$$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} = \frac{0.027027 - 0.052632}{64 - 8}$$
$$= -0.000457$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$
$$= \frac{-0.00347 - (-0.107143)}{27 - 0} = 0.00384$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$$
$$= \frac{-0.000457 - (-0.00347)}{64 - 1} = 0.000048$$

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0}$$
$$= \frac{0.000048 - 0.00384}{64 - 0} = -0.000059$$

x_i	$f(x_i)$	$1^{st}DD$	$2^{nd}DD$	$3^{rd}DD$	$4^{th}DD$
0	0				
		1			
1	1		−0.107143		
		0.142857		0.00384	
8	2		−0.00347		−0.000059
		0.052632		0.000048	
27	3		−0.000457		
		0.027027			
64	4				

To find $f(20)$, let

$$x_0 = 0, x_1 = 1, x_2 = 8, x_3 = 27, x_4 = 64$$

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

$$P_4(x) = 0 + (1)(x - 0) + (-0.107143)(x - 0)(x - 1) + (0.00384)(x - 0)(x - 1)(x - 8) + (-0.000059)(x - 0)(x - 1)(x - 8)(x - 27)$$

$$\therefore\ f(20) \approx P_4(20)$$
$$= (1)(20-0) + (-0.107143)(20-0)(20-1)$$
$$+ (0.00384)(20-0)(20-1)(20-8)$$
$$+ (-0.000059)(20-0)(20-1)(20-8)(20-27)$$

Please complete the solution…

Example 3:
Consider the data in the following table

x	-1	0	1	2
$f(x)$	3	-4	5	-6

use Newton's divided difference to estimate $f(-1.5)$.

Solution:

$$f[x_0] = f(x_0) = 3$$
$$f[x_1] = f(x_1) = -4$$
$$f[x_2] = f(x_2) = 5$$
$$f[x_3] = f(x_3) = -6$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{-4-3}{0-(-1)} = -7$$
$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{5-(-4)}{0-1} = 9$$
$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{-6-5}{2-1} = -11$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{9-(-7)}{1-(-1)} = 8$$
$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{-11-9}{2-0} = -10$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-10-8}{2-(-1)} = -6$$

x_i	$f(x_i)$	$1^{st}DD$	$2^{nd}DD$	$3^{rd}DD$
-1	3			
		-7		
0	-4		8	
		9		-6
1	5		-10	
		-11		
2	-6			

To find $f(1.5)$, let

$$x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2$$

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$P_3(x) = 3 + (-7)\big(x - (-1)\big) + (8)\big(x - (-1)\big)(x - 0) + (-6)\big(x - (-1)\big)(x - 0)(x - 1)$$

$$P_3(x) = -6x^3 + 8x^2 + 7x - 4$$

$$\therefore\ f(1.5) \approx P_3(x) = -6(-1.5)^3 + 8(-1.5)^2 + 7(-1.5) - 4$$

Please complete the solution…

***<u>Home work</u>*1: (H.W. 1):**

Given that

$f(-2) = 46, f(-1) = 4, f(3) = 156 \; and \; f(4) = 484,$
compute $f(0)$ by Newton's divided difference formula.

***<u>Home work</u>*2: (H.W. 2):**

(1) Complete the following divided difference table.

(2) Find the interpolating polynomial.

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \dots, x_i]$	$f[x_{i-4}, \dots, x_i]$
0	1.0	0.7651977				
1	1.3	0.6200860	-0.4837057			
2	1.6	0.4554022	-0.5489460	-0.0494433		
3	1.9					
4	2.2	0.1103623				