Gronwall Theorem:

If k is nonnegative constant, f, g are nonnegative fun.

$\alpha \le t \le \beta$ and $f(t) \le k + \int_\alpha^t f(s) \cdot g(s)\, ds$

then $\boxed{f(t) \le k\, e^{\int_\alpha^t g(s)\, ds}}$

Proof: Let $u(t) = k + \int_\alpha^t f(s) \cdot g(s)\, ds$

$f(t) \le u(t)$

$u(\alpha) = k$

$u'(t) = f(t) \cdot g(t)$

if $g(t) \ge 0$

$u'(t) \le u(t)\, g(t)$

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$$e^{-\int_\tau^t g(s)\,ds}$$

$$u'(t)\,e^{-\int_\tau^t g(s)\,ds} - u(t)\,g(t)\,e^{-\int_\tau^t g(s)\,ds} \leq 0$$

$$\leq u(t)\,e^{-\int_\tau^t g(s)\,ds}$$

then $\dfrac{d}{dt}\left[u(t)\,e^{-\int_\tau^t g(s)\,ds}\right] \leq 0$

$$u(t)\,e^{-\int_\tau^t g(s)\,ds} - u(\tau) \leq 0$$

$$f(t) \leq u(t)$$

$$\therefore f(t) \leq u(t) \leq u\,e^{\int_\tau^t g(s)\,ds}$$

Ex: $y' = -y$, $y(0) = 1 \leftarrow y_0$

Sol: $f(t,y) = -y$

$$\phi_j(t) = y_0 + \int_{t_0}^{t} f(s, \phi_{j-1}(s))\,ds$$

$j=1$

$$\phi_1(t) = y_0 + \int_0^t f(s, \phi_0(s))\,ds$$

$$= 1 + \int_0^t -1\,ds = 1 - t$$

$j=2$

$$\phi_2(t) = y_0 + \int_0^t f(S, \phi_1(s))\,d$$

$$= 1 + \int_0^t -(1-S)\,ds = 1 - \left(t - \frac{t^2}{2}\right) = 1 - t + \frac{t^2}{2}$$

$$j=3 \Rightarrow 1 + \int_0^t -\left(1 - s - \frac{s^2}{2}\right) ds$$

$$= 1 + \left[-s + \frac{s^2}{2} + \frac{s^3}{6}\right]_0^t$$

$$= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!})$$

$$\Rightarrow 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \cdots\cdots$$

$$\simeq e^{-t}$$