Foundation of Mathematics I

Chapter 2   Sets

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Chapter Two
Sets

2.1. Definitions

Definition 2.1.1. A **set** is a collection of (objects) things. The things in the collection are called **elements (member)** of the set.

A set with no elements is called **empty set** and denoted by $\emptyset$; that is, $\emptyset = \{\}$.
A set that has only one element, such as $\{x\}$, is sometimes called a **singleton set**.

List of the symbols we will be using to define other terminologies:
$|$ **or :** : such that
$\in$: an element of (belong to)
$\notin$: not an element of (not belong to)
$\subset$ **or** $\subsetneq$: a proper subset of
$\subseteq$: a subset of
$\nsubseteq$: not a subset of
$\mathbb{N}$: Set of all natural numbers
$\mathbb{Z}$: Set of all integer numbers
$\mathbb{Z}^+$: Set of all positive integer numbers
$\mathbb{Z}^-$: Set of all negative integer numbers
$\mathbb{Z}_o$: Set of all odd numbers
$\mathbb{Z}_e$: Set of all even numbers
$\mathbb{Q}$: Set of all rational numbers
$\mathbb{R}$: Set of all real numbers

Set Descriptions 2.1.2.

(i) Tabulation Method
The elements of the set listed between commas, enclosed by braces.
(1) $\{1,2,37,88,0\}$
(2) $\{a, e, i, o, u\}$ Consists of the lowercase vowels in the English alphabet.
(3) $\{\dots, -4, -2, 0, 2, 4, 6\}$ Continue from left side
$\{-4, -2, 0, 2, 4, 6, \dots\}$ Continue from right side
$\{\dots, -4, -2, 0, 2, 4, 6, \dots\}$ Continue from left and right sides.
(4) $B = \{\{2,4,6\}, \{1,3,7\}\}$.

(ii) Rule Method
Describe the elements of the set by listing their properties writing as

$$S = \{x \mid A(x)\},$$

where $A(x)$ is a statement related to the elements x. Therefore,

$$x \in S \Leftrightarrow A(x) \text{ is hold}$$

(1) $A = \{x \mid x \text{ is a positive integers and } x > 10\}$
$A = \{x \mid x \in \mathbb{Z}^{+} \text{and } x > 10\}$.
(2) $\mathbb{Z}_o = \{x \mid x = 2n - 1 \text{ and } n \in \mathbb{Z}\}$
$= \{2n - 1 \mid n \in \mathbb{Z}\}$.
(3) $\{x \in \mathbb{Z} \mid |x| < 4\} = \{-3, -2, -1, 0, 1, 2, 3\}$.
(4) $\{x \in \mathbb{Z} \mid x^2 - 2 = 0\} = \emptyset$.

Examples 2.1.3.
(i) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ Integer numbers.
(ii) $\mathbb{Z}_e = \{x \mid x = 2n \text{ and } n \in \mathbb{Z}\}$
$= \{2n \mid n \in \mathbb{Z}\}$. Even numbers
Note that 2 is an element of $\mathbb{Z}_e$ so, we write $2 \in \mathbb{Z}$. But, $5 \notin \mathbb{Z}_e$.

(iii) Let C be the set of all natural numbers which are less than 0.
In this set, we observe that there are no elements. Hence, C is an empty set; that is,

$$C = \emptyset.$$

Definition 2.1.4.
(i) A set A is said to be a **subset** of a set B if every element of A is an element of B and denote that by $A \subseteq B$. Therefore,

$$A \subseteq B \Leftrightarrow \forall x (x \in A \Longrightarrow x \in B).$$

(ii) If A is a nonempty subset of set B and B contains an element which is not a member of A, then A is said to be **proper subset** of B and denoted this by $A \subset B$ or $A \subsetneq B$; that is, A is said to be a **proper subset** of B if and only if
(1) $A \neq \emptyset$, (2) $A \subset B$ and (2) $A \neq B$.

We use the expression $A \nsubseteq B$ means that A is **not** a subset of B.

Examples 2.1.5.
(i) An empty set $\emptyset$ is a subset of any set B; that is, for every set B, $\phi \subseteq B$.

If this were not so, there would be some element $x \in \emptyset$ such that $x \notin B$. However, this would contradict with the definition of an empty set as a set with no elements.

(ii) Let B be the set of natural numbers. Let A be the set of even natural numbers. Clearly, A is a subset of B. However, B is not a subset of A, for $3 \in B$, but $3 \notin A$.

Theorem 2.1.6. (Properties of Sets)
Let A, B, and C be sets.
(i) For any set $A, A \subseteq A$. (Reflexive Property)
(ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (Transitive Property)

Proof.
(ii)

1	$(A \subseteq B) \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$	Hypothesis and Def. $\subseteq$
2	$(B \subseteq C) \Leftrightarrow \forall x(x \in B \Rightarrow x \in C)$	Hypothesis and Def. $\subseteq$
	$\Rightarrow \forall x(x \in A \Rightarrow x \in C)$	Inf. (1),(2) Syllogism Law
	$\Leftrightarrow A \subseteq C$	Def. of $\subseteq$

Definition 2.1.7 If X is a set, the **power set** of X is another set, denoted as $P(X)$ and defined to be the set of all subsets of X. In symbols,

$$P(X) = \{A | A \subseteq X\}.$$

That is, $A \subseteq X$ if and only if $A \in P(X)$.

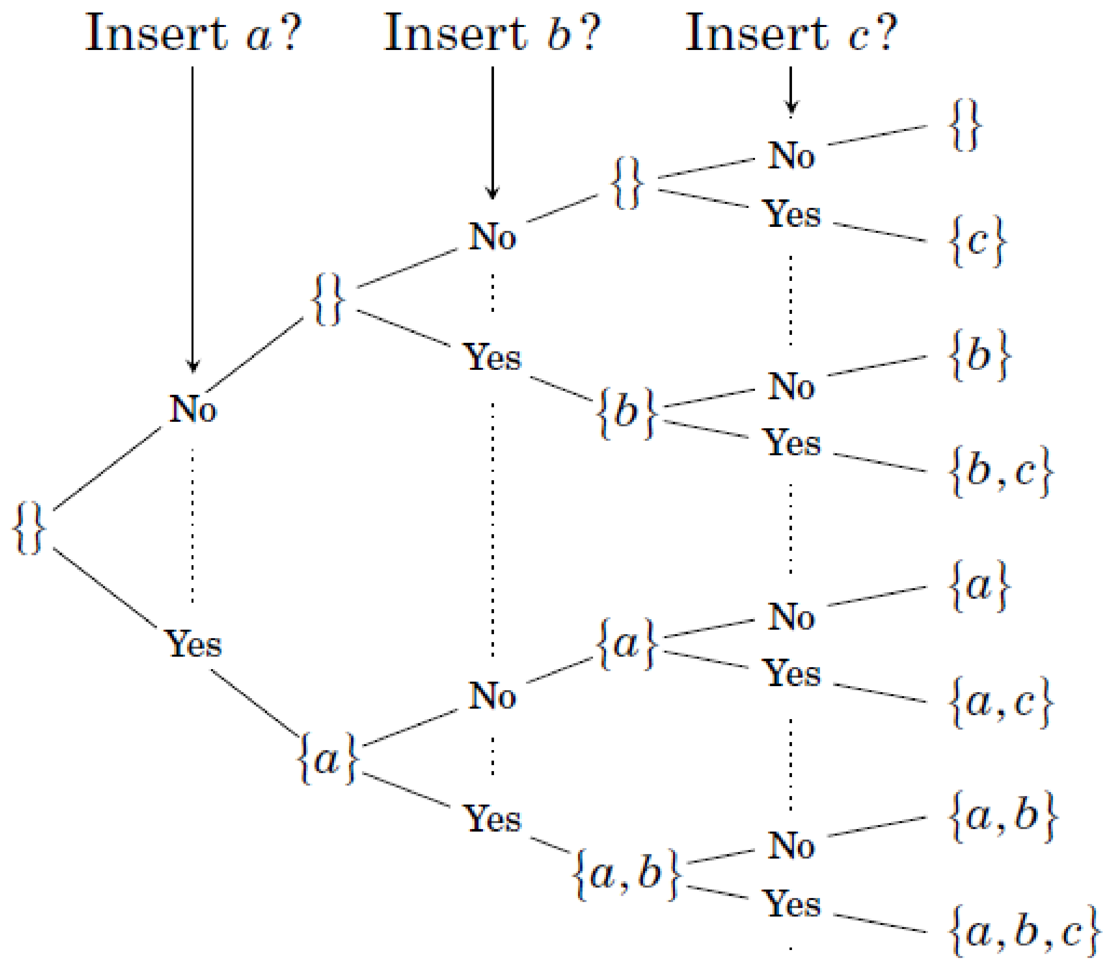
Example 2.1.8.

(i) $\emptyset$ and a set X are always members of $P(X)$.

(ii) suppose $X = \{a, b, c\}$. Then

$$P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

The way to finding all subsets of X is illustrated in the following figure.



From the above example, if a finite set X has n elements, then it has $\mathbf{2^n}$ subsets, and thus its power set has $\mathbf{2^n}$ elements.

(iii) $P(\{1,2,4\}) = \{\emptyset, \{0\}, \{1\}, \{4\}, \{0,1\}, \{0,4\}, \{1,4\}, \{1,2,4\}\}$.
(iv) $P(\emptyset) = \{\emptyset\}$.
(v) $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
(vi) $P(\{\mathbb{Z}, \mathbb{R}\}) = \{\emptyset, \{\mathbb{Z}\}, \{\mathbb{R}\}, \{\mathbb{Z}, \mathbb{R}\}\}$.

The following are wrong statements.

(v) $P(1) = \{\emptyset, \{1\}\}$.
(vi) $P(\{1, \{1,2\}\}) = \{\, \emptyset, \{1\}, \{1,2\}, \{1, \{1,2\}\} \,\}$.
(vii) $P(\{1, \{1,2\}\}) = \{\, \emptyset, \{\{1\}\}, \{\{1,2\}\}, \{1, \{1,2\}\} \,\}$.