$$f(x)=\begin{cases}0 & x\in G_1\\ 1 & x\in G_2\end{cases}$$

$\Rightarrow f$ is onto. T.p f is conts

$\because$ $\{0,1\}$ has the discrete topology $\Rightarrow \{0\},\{1\},\phi,\{0,1\}$ are open sets in $\{0,1\}$. Since $G_1\cap G_2=\phi$, then

$f^{-1}(\{0\})=G_1\in\tau$, $f^{-1}(\{1\})=G_2\in\tau$, $f^{-1}(\phi)=\phi\in\tau$ & $f^{-1}(\{0,1\})=X\in\tau$

$\Rightarrow f$ is continuous.

Conversely suppose that $\exists$ a continuous function f from (X,τ) onto the discrete two point space $\{0,1\}$.

T.p (X,τ) is disconnected.

Suppose (X,τ) is connected $\xrightarrow{\text{By Theorem}}$ $\{0,1\}$ is connected C (since every discrete space contains more than one point is disconnected). Therefore (X,τ) is disconnected.

.24 Theorem: A topological space (X,τ) is connected iff every continuous function from X into the discrete two points space is constant.

$\{y\}=f(x)$

$x=f^{-1}(f(x))=\phi$

proof: Let (X,τ) be a connected space & $f:(X,\tau)\to(\{0,1\},D)$ be any conts function. To prove f is constant

Let $y\in f(X)\subset\{0,1\}\Rightarrow\{y\}$ is both open & closed in $(\{0,1\},D)$

$\because f$ is conts $\Rightarrow f^{-1}(\{y\})$ is closed, open and non-empty.

Since (X,τ) is connected $\Rightarrow f^{-1}(\{y\})=X\Rightarrow f$ is constant

Conversely, suppose that every conts function from (X,τ) into the discrete space $\{0,1\}$ is constant. To prove (X,τ) is connected

suppose that (X,τ) is disconnected $\Rightarrow\exists A\subseteq X$ s.t

$\phi\neq A\neq X$ & A is both open & closed in $X\Rightarrow\phi\neq A^c\neq X$ &

~~A^c is both open & closed in X. Now consider the characteristic~~ function χ_A of A defined by:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$$

$\therefore \chi_A^{-1}(\phi) = \phi$, $\chi_A^{-1}(\{1\}) = A$, $\chi_A^{-1}(\{0\}) = A^c$ & $\chi_A^{-1}(\{0,1\}) = X$ are open in $X \Rightarrow \chi_A$ is conts. But χ_A is not constant C!

Hence (X,τ) is a connected space.

5.25
Questions: ① The continuous image of a connected set is connected.

② Show that a topological space (X,τ) is connected iff there exists no continuous function from X onto the discrete two points space $\{0,1\}$.

Sol ①
Let $f:(X,\tau) \longrightarrow (Y,\tau')$ be a conts function & H be a connected in X T.P $f(H)$ is connected in Y.
Suppose that $f(H)$ is disconnected $\Rightarrow \exists\, G_1, G_2 \underset{open}{\subseteq} Y$ s.t

$f(H) \cap G_1 \neq \phi$, $f(H) \cap G_2 \neq \phi$; $(f(H) \cap G_1) \cap (f(H) \cap G_2) = \phi$

& $(f(H) \cap G_1) \cup (f(H) \cap G_2) = f(H)$

$\therefore f(H) \cap G_1 \neq \phi \Rightarrow \exists\, h \in H$ s.t $f(h) \in G_1 \Rightarrow h \in f^{-1}(G_1)$
$\Rightarrow H \cap f^{-1}(G_1) \neq \phi$,
Similarly, $H \cap f^{-1}(G_2) \neq \phi$

$\therefore f$ is conts $\Rightarrow f^{-1}(G_1), f^{-1}(G_2) \underset{open}{\subseteq} X \Rightarrow H \cap f^{-1}(G_1), H \cap f^{-1}(G_2) \underset{open}{\subseteq} H$

$\because (f(H)\cap G_1)\cap(f(H)\cap G_2)=\phi \Rightarrow f(H)\cap G_1\cap G_2=\phi$

$\Rightarrow H\cap f^{-1}(G_1)\cap f^{-1}(G_2)\subseteq f^{-1}(f(H))\cap f^{-1}(G_1)\cap f^{-1}(G_2)=f^{-1}(\phi)=\phi$

$\Rightarrow H\cap f^{-1}(G_1)\cap f^{-1}(G_2)=\phi \Rightarrow (H\cap f^{-1}(G_1))\cap(H\cap f^{-1}(G_2))=\phi$

Also, $(f(H)\cap G_1)\cup(f(H)\cap G_2)=f(H)$

$\Rightarrow f(H)\cap(G_1\cup G_2)=f(H) \Rightarrow f(H)\subseteq G_1\cup G_2$

$\Rightarrow H\subseteq f^{-1}(f(H))\subseteq f^{-1}(G_1\cup G_2)$

$\Rightarrow H\cap f^{-1}(G_1\cup G_2)=H \Rightarrow (H\cap f^{-1}(G_1))\cup(H\cap f^{-1}(G_2))=H$

$\Rightarrow$ H is disconnected in X. C!

$\therefore f(H)$ is connected in Y.