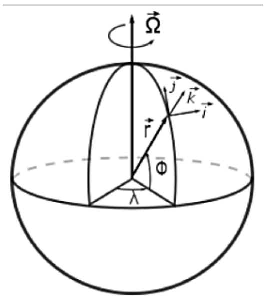
Lecture 4

The Spherical Coordinates

4.1 Momentum Equations in Spherical Coordinates

The equation of motion (and the components forms) which we have taken in the previous lectures are to be applied for Cartesian coordinates. However, our earth is a globe and we should consider the spherical coordinates instead. Hence, it is convenient to expand the momentum equation(s) of motion in spherical coordinates. The coordinate axes are then (λ, ϕ, z), where λ is longitude, ϕ is latitude, and z is the vertical distance above the surface. If the unit vectors i, j, and k are now taken to be directed eastward, northward, and upward, respectively. Then:



$$\vec{V} \equiv iu + jv + kw$$

where $u, v \; and \; w$ are defined as:

$$u \equiv r\cos\phi \, \frac{d\lambda}{dt} \quad , \quad v \equiv r\, \frac{d\phi}{dt} \quad , \quad w \equiv \frac{dz}{dt} \qquad (4.1)$$

Here, r is the distance to the center of earth, which is related to z by $= a + z$, where a is the radius of the earth. Traditionally, the variable r in (4.1) is replaced by the constant a for notational simplicity (Why?).

It is conventional to define x and y as eastward and northward distance such that:

$$dx = a\cos\phi \, d\lambda \qquad and \qquad dy = a\, d\phi$$

Thus, $u \; \equiv \frac{dx}{dt} \quad and \; v \equiv \frac{dy}{dt}$ are the horizontal velocity components.

The (x, y, z) coordinate system defined in this way is not a Cartesian coordinate system because the directions of the i,j,k unit vectors are not constant as function of position on the spherical earth (since they rotate with Earth). Thus:

$$\frac{d\vec{V}}{dt} = i\frac{du}{dt} + j\frac{dv}{dt} + k\frac{dw}{dt} + u\frac{di}{dt} + v\frac{dj}{dt} + w\frac{dk}{dt} \qquad (4.2)$$

Note that we differentiate the components of velocity (u, v, and w) and also the unit vectors (i, j, and k).

Now, we first consider $\frac{di}{dt}$ and by expanding the total derivative and noting that i is a function only of x:

$$\boxed{\frac{di}{dt} = \frac{\partial i}{\partial t} + u\frac{\partial i}{\partial x} + v\frac{\partial i}{\partial y} + w\frac{\partial i}{\partial z}}$$

$$\boxed{\begin{aligned} dx &= a\cos\phi\, d\lambda \\ \frac{d\lambda}{dx} &= \frac{1}{a\cos\phi} \end{aligned}}$$

$$\frac{di}{dt} = u\frac{\partial i}{\partial x}$$

From Figure (4.1), we see by similarity of triangles:

$$\lim_{\delta x \to 0} \frac{|\delta i|}{\delta x} = \left|\frac{\partial i}{\partial x}\right| = \frac{1}{a\cos\phi}$$

and that the vector $\frac{\partial i}{\partial x}$ is directed toward the axis of rotation. Thus, as is illustrated in Fig (4.2):

$$\frac{\partial i}{\partial x} = \frac{1}{a\cos\phi}(j\sin\phi - k\cos\phi)$$

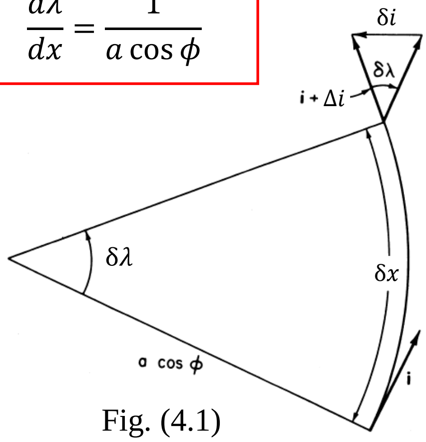


Fig. (4.1)

Therefore:

$$\frac{di}{dt} = \frac{u}{a\cos\phi}(j\sin\phi - k\cos\phi) \qquad (4.3)$$

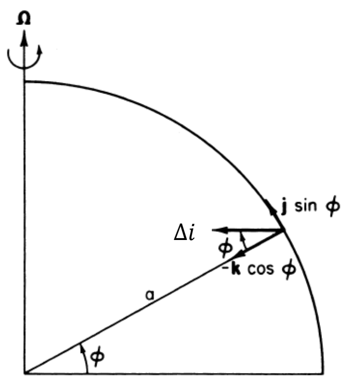


Fig. (4.2) Resolution of Δi in Fig. 9.1 into northward and vertical components

Considering now $\frac{dj}{dt}$, we note that j is a function only of x and y. Thus, with the aid of Fig. (4.3) we see that for eastward motion $|\delta j| = \delta x/(a/\tan\phi)$.

Because the vector $\frac{dj}{dx}$ is directed in the negative x direction, We have then:

$$\frac{dj}{dx} = -\frac{\tan\phi}{a}i$$

From Fig. 4.4 it is clear that for northward motion $|\delta j| = \delta\phi$, and $\delta y = a\,\delta\phi$ and δj is directed downward so that:

$$\frac{dj}{dy} = -\frac{k}{a}$$

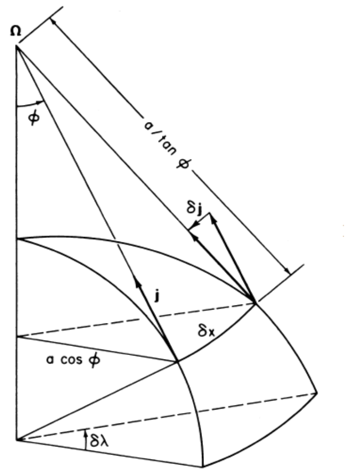


Fig. (4.3) Dependence of j on longitude

Hence,

$$\frac{\mathrm{d}j}{dt} = -\frac{u\tan\phi}{a}i - \frac{v}{a}k \qquad (4.4)$$

Finally, by similar arguments it can be shown that:

$$\frac{\mathrm{d}k}{dt} = i\frac{u}{a} + j\frac{v}{\bar{a}} \qquad (4.5)$$

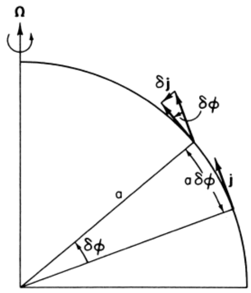


Fig. (4.4) Dependence of *j* on latitude

Substituting (4.3)-(4.5) into (4.2) and rearranging the terms,

We obtain the spherical polar coordinate expansion of the acceleration following the relative motion:

$$\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a}\right)i + \left(\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a}\right)j + \left(\frac{dw}{dt} - \frac{u^2+v^2}{a}\right)k \qquad (4.6)$$

From Equation (3.1) in Lecture (3) (please see the equation in components form not in vector):

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla \mathrm{p} + \vec{g} + \vec{\mathrm{F}}_r - 2\vec{\Omega} \times \vec{V}$$

By substituting the left side from equation (4.6) and the right side terms from lecture (3) and decomposed the equation into the three directions:

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + 2\Omega w\cos\phi - 2\Omega v\sin\phi \qquad (4.7)$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) - 2\Omega u\sin\phi \qquad (4.8)$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) - g + 2\Omega u\cos\phi\, k \qquad (4.9)$$

Notes:

1. The terms proportional to 1/a on the left side of momentum equations above are called curvature terms; they arise due to the curvature of the earth. Because they are nonlinear (i.e., they are quadratic in the dependent variable), they are difficult to handle in theoretical analyses. Fortunately, the curvature terms are unimportant for midlatitude synoptic scale motions.
2. Even if we neglect the curvature terms, the equations still nonlinear. This is obvious when we expand the total derivative ($\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$).
3. The presence of nonlinear advection processes is one reason that dynamic meteorology is an interesting and challenging subject.