Lecture 6

The Continuity Equation (the conservation of mass)

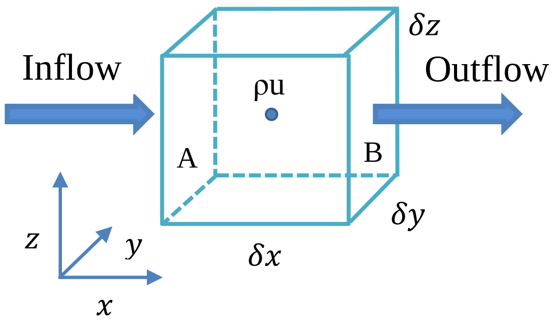
6.1 Introduction

Mass is not created or destroyed – it is always conserved. This idea is often termed continuity. However، even if the mass remains constant, the volume may change. Air can expand or flow outward (a process called divergence) or compress (convergence). Hence, mass conservation requires considering how the density changes in the presence of divergence or convergence to keep the total mass constant.

6.2 Derivation of continuity equation

Continuity equation is one of the seven equations controlling the atmospheric motion and we have to derive it and describe it in three dimensions.

Consider a small volume of air ($Vol = \delta x \delta y \delta z$) at some fixed point in our Eulerian frame of reference. The mass of air in that volume at any instant is simply the density multiplied by the volume ($m = Vol \times \rho$).



A simple continuity equation is:

$$\frac{\partial M}{\partial t} = \text{Inflow rate} - \text{Outflow rate}$$

The x-direction **mass flux** (i.e. the product of x-direction velocity and the density of the fluid) at the center of the cube is given by ρu. If we expand this function in a Taylor series about the center point, we find that the rate of mass <u>inflow</u> through side A of the cube is given by:

$$\left[\rho\, u - \frac{\partial}{\partial x}(\rho\, u)\frac{\delta x}{2}\right]\delta y \delta z$$

while the rate of mass outflow through side B of the cube is given by:

$$\left[\rho\, u + \frac{\partial}{\partial x}(\rho\, u)\frac{\delta x}{2}\right]\delta y \delta z$$

the rate of accumulation of mass:

$$\frac{\partial M_x}{\partial t} = \text{Inflow rate} - \text{Outflow rate}$$

$$\frac{\partial M_x}{\partial t} = \left[\rho\, u - \frac{\partial}{\partial x}(\rho\, u)\frac{\delta x}{2}\right]\delta y \delta z - \left[\rho\, u + \frac{\partial}{\partial x}(\rho\, u)\frac{\delta x}{2}\right]\delta y \delta z$$

$$\frac{\partial M_x}{\partial t} = -\frac{\partial}{\partial x}(\rho\, u)\delta x \delta y \delta z$$

in similarity,

$$\frac{\partial M_y}{\partial t} = -\frac{\partial}{\partial y}(\rho\, v)\delta x \delta y \delta z \qquad \text{and} \qquad \frac{\partial M_z}{\partial t} = -\frac{\partial}{\partial z}(\rho\, w)\delta x \delta y \delta z$$

so that the net rate of mass accumulation in the cube is represented as:

$$\frac{\partial M}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w)\right]\delta x \delta y \delta z$$

dividing by the volume yields:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w)\right] = -\nabla.(\rho\, \vec{V}) \qquad (6.1)$$

where $\vec{V}$ is the velocity vector. The above expression is called the **mass divergence** form of the mass continuity equation.

An alternative form of this expression arises by recalling that:

$$-\nabla.\left(\rho\, \vec{V}\right) = \rho \nabla.\vec{V} + \vec{V}.\nabla\rho$$

so that eq. (6.1) becomes:

$$\frac{\partial \rho}{\partial t} + \vec{V}.\nabla\rho + \rho\nabla.\vec{V} = 0 \qquad or \qquad \frac{1}{\rho}\frac{d\rho}{dt} + \nabla.\vec{V} = 0 \qquad (6.2)$$

which is known as the **velocity divergence** form of the mass continuity equation.

This exact same relationship can be derived for a cube of fixed mass, $\delta M,$ but varying dimensions δx , $\delta y,$ and $\delta z.$

Given that the mass in this example is fixed, then $d(\delta M)/dt = 0$ or by the chain rule:

$$\frac{d(\rho\delta x\delta y\delta z)}{dt} = 0 = \frac{d\rho}{dt}\delta x\delta y\delta z + \rho\frac{d(\delta x)}{dt}\delta y\delta z + \rho\frac{d(\delta y)}{dt}\delta x\delta z + \rho\frac{d(\delta z)}{dt}\delta x\delta y$$

Now

$$\lim_{\delta x\to 0}\frac{d(\delta x)}{dt} = \partial u$$

with similar expressions applying for the last two time derivatives and dividing both sides by volume gives:

$$\frac{d\rho}{dt} + \rho\frac{\partial u}{dx} + \rho\frac{\partial v}{dy} + \rho\frac{\partial w}{dz} = \frac{d\rho}{dt} + \rho\nabla.\vec{V} = 0 \qquad (6.3)$$

which can be easily rearranged into (6.2).

6.3 Comprisable and incompressible fluids

A fluid in which individual parcels experience no change of density following the motion (i.e. $\frac{d\rho}{dt} = 0$) is known as an incompressible fluid. Conversely، a compressible fluid is one in which the density can change along a parcel trajectory. As you might guess, the atmosphere is a compressible fluid, but for many atmospheric phenomena, the compressibility is not of physical importance.

In such cases, the mass continuity equation becomes a statement of zero velocity divergence($\frac{d\rho}{dt} + \rho\nabla.\vec{V} = 0 \;\rightarrow\rightarrow\; 0 + \rho\nabla.\vec{V} = 0 \;\rightarrow\rightarrow\; \nabla.\vec{V} = 0$). We will see later that the non-divergence idea is very important in many aspects.

Note: the **Lagrangian flow** is a way of looking at fluid motion where the observer follows an individual fluid parcel as it moves through space and time. This can be visualized as sitting in a boat and drifting down a river.

The **Eulerian flow** is a way of looking at fluid motion that focuses on specific locations in the space through which the fluid flows as time passes. This can be visualized by sitting on the bank of a river and watching the water pass the fixed location.

The Lagrangian and Eulerian flows are sometimes denoted as the **Lagrangian and Eulerian frame of reference**.