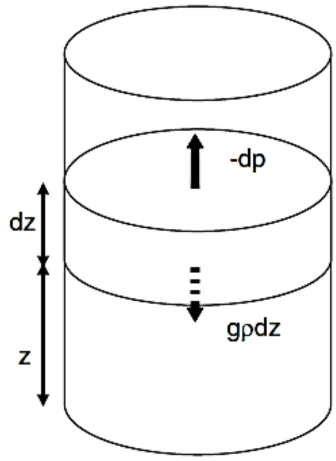
Lecture 7

The Static Equilibrium in the Atmosphere

7.1 The Results of Hydrostatic Balance

Newton's law requires that the upward force acting on a thin layer of air from the decrease of pressure with height is generally closely balanced by the downward force due to gravity (as in the figure). The hydrostatic equation is then:

$$\frac{\partial p}{\partial z} = -g\rho$$



Typically, deviations from the hydrostatic balance occur locally, e.g. in updrafts and downdrafts of storms or when the air hits a small obstacle. In this case, the air particle undergoes acceleration:

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{dp}{dz} - g$$

By using the Hydrostatic Balance, we are going to discuss two typical models that approximate the atmosphere:

A. The Homogeneous Atmosphere

In this atmosphere, the density is considered constant anywhere (spatially constant)

$\rho = \rho_o = constant$ (where ρ_o is the air density at the surface?)

From the hydrostatic equation $\frac{dp}{dz} = -g\rho_o$

$$\int_{p_o}^{p} dp = -g\rho_o \int_0^z dz$$

$$p - p_o = -g\,\rho_o\,z$$

$$p = p_o - g\rho_o z \qquad (7.1)$$

Where p_o is pressure at $z = 0$

The homogeneous atmosphere has a finite height H,

when p=0 (at the top of the atmosphere), z=H

$$0 = p_o - \rho_o g H$$

$$\therefore H = \frac{p_o}{\rho_o g}$$

And from the hydrostatic equation and the equation of state ($p_o = \rho_o\, R\, T_o$) we get:

$$H = \frac{\rho_o\, R\, T_o}{\rho_o\, g}$$

At $T_o = 283^o K\,,\ R = 287\,,\ \ g = 9.8\, ms^{-2} \Rightarrow H \approx 8000\, m$

(Homework: solve for T_o = 293, 300, 310, 320, 330 ^{o}K)

- We may define a temperature in the homogeneous atmosphere from gas equation:

$$p = \rho_o R\, T$$

$$T = \frac{p}{\rho_o R} \qquad (7.2)$$

Put eqn. (7.1) in eqn. (7.2) you get:

$$T = \frac{p_o - \rho_o\, g\, z}{\rho_o\, R} \quad \Rightarrow \quad T = \frac{p_o}{\rho_o R} - \frac{\rho_o\, g\, z}{\rho_o R}$$

$$\boxed{T = T_o - \frac{g}{R} z}$$

This equation shows that T decreases linearly with height in a homogeneous atmosphere (*note the -ve sign*)

Question: From the homogeneous atmosphere model show that the lapse rate $\gamma = \frac{dT}{dZ} = -\frac{g}{R} = -3.4\ \ ^oK/100m$

B. The Isothermal Atmosphere

In this model we have $T = T_o = const.$ (where T_o is the temperature at the surface)

From the hydrostatic Equation we get:

$$dp = -\rho g\, dz$$

Recall that: $\rho = \frac{p}{RT_o}$

$$dp = \frac{p}{RT_o} g\, dz$$

$$\int_{p_o}^{p} \frac{dp}{p} = -\frac{g}{RT_o} \int_0^z dz$$

$$\ln \frac{p}{p_o} = -\frac{g}{RT_o} z$$

Taking exponential to both sides

$$\frac{p}{p_o} = e^{-\frac{g}{RT_o} z}$$

This equation shows that the isothermal atmosphere is of infinite extent because $p \to 0$ when $z \to \infty$

$$\boxed{p = p_o e^{-\frac{g}{RT_o} Z}}$$

The **scale height** for an isothermal atmosphere is often defined as the height at which the pressure has decreased to e^{-1} of the surface pressure.

$$z = H_s$$

$$p = p_o \, e^{-\frac{g}{RT_o} H_s}$$

$$p = p_o \, e^{-1}$$

$$p_o e^{-\frac{g}{R\,T_o} H_s} = p_o \, e^{-1}$$

$$-\frac{g}{RT_o} H_s = -1$$

$$\therefore H_s = \frac{R\,T_o}{g} = 8000 \, m$$

Or, that the **scale height** is equal to the **height of the homogeneous atmosphere** having the same surface temperature as the isothermal atmosphere.

The density in the isothermal atmosphere can be calculated from gas equation

$p_o = \rho_o \, R \, T_o$, $p = \rho \, R \, T_o$:

$$p = p_o e^{-\frac{g}{RT_o} Z}$$

$$\rho \, R \, T_o = \rho_o \, R \, T_o \, e^{-\frac{g}{RT_o} z}$$

$$\therefore \rho = \rho_o \, e^{-\frac{g}{RT_o} z}$$

Problem 1: Show that a homogeneous atmosphere (density independent of height) has a finite height that depends only on the temperature at the lower boundary. Compute the height of a homogeneous atmosphere with surface temperature T_0 = 273K and surface pressure 1000 hPa. (Use the ideal gas law and hydrostatic balance.)

Problem 2: Show that in an atmosphere with uniform lapse rate γ (where $\gamma \equiv -\frac{dT}{dz}$) the geopotential height at pressure level p_1 is given by

$$Z = \frac{T_0}{\gamma}\left[1 - \left(\frac{p_0}{p_1}\right)^{-R\gamma/g}\right]$$

where T_0 and p_0 are the sea level temperature and pressure, respectively.

(Hint: Use the hydrostatic equation and the ideal gas law.)