GRAPHS 3

3.1 WHAT IS A GRAPH ? DEFINITION

A graph G consists of a set of objects V = {v_1, v_2, v_3, ...} called **vertices** (also called **points** or **nodes**) and other set E = {e_1, e_2, e_3, ...} whose elements are called **edges** (also called **lines** or **arcs**).

The set V(G) is called the **vertex set** of G and E(G) is the **edge set.**

Usually the graph is denoted as **G = (V, E)**

Let G be a graph and {*u*, *v*} an edge of G. Since {*u*, *v*} is 2-element set, we may write {*v*, *u*} instead of {*u*, *v*}. It is often more convenient to represent this edge by *uv* or *vu.*

If *e* = *uv* is an edge of a graph G, then we say that *u* and *v* are **adjacent** in G and that *e* joins *u* and *v*. (We may also say that each that of *u* and *v* is adjacent to or with the other).

For example :

A graph G is defined by the sets

V(G) = {*u, v, w, x, y, z*} and E(G) = {*uv, uw, wx, xy, xz*}.

Now we have the following graph by considering these sets.

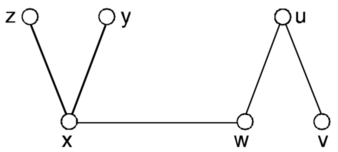


Fig. 3.1

Every graph has a diagram associated with it. The vertex *u* and an edge *e* are **incident** with each other as are *v* and *e*. If two distinct edges say *e* and *f* are **incident** with a common vertex, then they are adjacent edges.

A graph with *p*-vertices and *q*-edges is called a **(*p, q*) graph.**

The (1, 0) graph is called **trivial graph.**

In the following figure the vertices *a* and *b* are adjacent but *a* and *c* are not. The edges *x* and *y* are adjacent but *x* and *z* are not.

Although the edges *x* and *z* intersect in the diagram, their intersection is not a vertex of the graph.

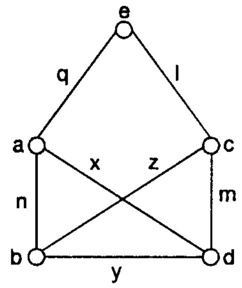


Fig. 3.2

Examples :

(1) Let V = {1, 2, 3, 4} and E = {{1, 2}, {1, 3}, {3, 2}. {4, 4}}.
Then G(V, E) is a graph.

(2) Let V = {1, 2, 3, 4} and E = {{1, 5}, {2, 3}}.
Then G(V, E) is not a graph, as 5 is not in V.

(3)

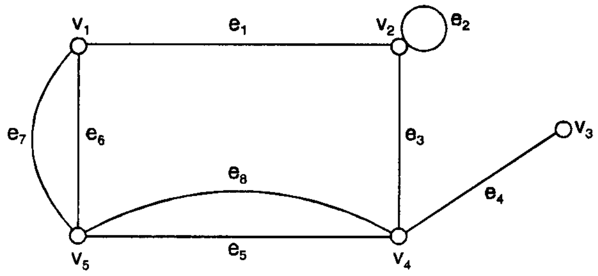


Fig. 3.3

A graph with 5-vertices and 8-edges is called a **(5, 8) graph.**

3.2 DIRECTED AND UNDIRECTED GRAPHS

3.2.1. Directed graph

A directed graph or digraph G consists of a set V of vertices and a set E of edges such that $e \in$ E is associated with an ordered pair of vertices.

In other words, if each edge of the graph G has a direction then the graph is called **directed graph.**

In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v.

Figure 3.4(a) is an example of a directed graph.

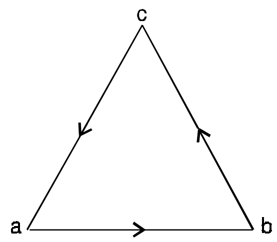


Fig. 3.4(*a*). Directed graph.

Suppose $e = (u, v)$ is a directed edge in a digraph, then (*i*) u is called the **initial vertex** of e and v is the terminal vertex of e

(*ii*) e is said to be **incident** from u and to be incident to v.

(*iii*) u is adjacent to v, and v is adjacent from u.

3.2.2. Un-directed graph

An un-directed graph G consists of set V of vertices and a set E of edges such that each edge $e \in$ E is associated with an unordered pair of vertices.

In other words, if each edge of the graph G has no direction then the graph is called **un-directed graph.**

Figure 3.4(*b*) is an example of an undirected graph.

We can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

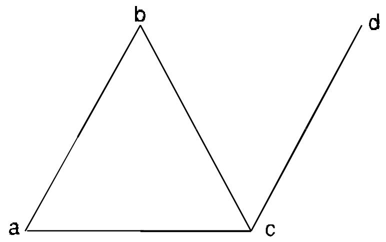


Figure 3.4(*b*). Un-directed graph.

3.3 BASIC TERMINOLOGIES

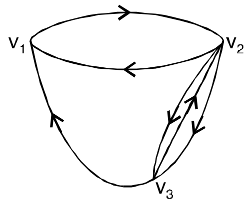
3.3.1 Loop : An edge of a graph that joins a node to itself is called **loop** or **self loop.**

i.e., a loop is an edge (v_i, v_j) where $v_i = v_f$.

3.3.2. Multigraph

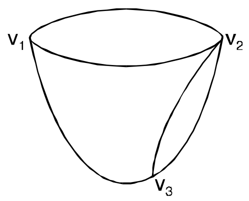
In a multigraph no loops are allowed but more than one edge can join two vertices, these edges are called **multiple edges** or parallel edges and a graph is called **multigraph.**

Two edges (v_i, v_j) and (v_f, v_r) are parallel edges if $v_i = v_r$ and v_j, v_f.



Directed multigraph

Fig. 3.5(*a*)



Un-directed multigraph

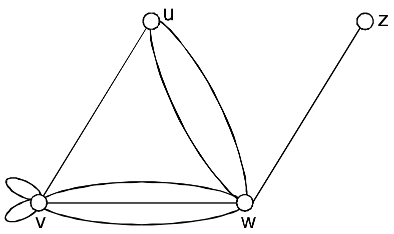
Fig. 3.5(*b*)

In Figure 3.5(*a*), there are two parallel edges associated with v_2 and v_3.

In Figure 3.5(*b*), there are two parallel edges joining nodes v_1 and v_2 and v_2 and v_3.

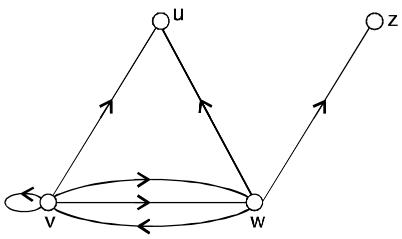
3.3.3. Pseudo graph

A graph in which loops and multiple edges are allowed, is called a pseudo graph.



Un-directed Pseudo graph

Fig. 3.6(*a*)



Directed Pseudo graph

Fig. 3.6(*b*)

3.3.4. Simple graph

A graph which has neither loops nor multiple edges. *i.e.,* where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph.**

Figure 3.4(*a*) and (*b*) represents simple undirected and directed graph because the graphs do not contain loops and the edges are all distinct.

3.3.5. Finite and Infinite graphs

A graph with finite number of vertices as well as a finite number of edges is called a **finite graph.** Otherwise, it is an **infinite graph.**

3.4 DEGREE OF A VERTEX

The number of edges incident on a vertex v_i with **self-loops counted twice** (is called the **degree of a vertex** v_i and is denoted by $\mathbf{deg_G(}\boldsymbol{v_i}\mathbf{)}$ or **deg** $\boldsymbol{v_i}$ or $\boldsymbol{d(v_i)}$**.**

The degrees of vertices in the graph G and H are shown in Figure 3.7(*a*) and 3.7(*b*).

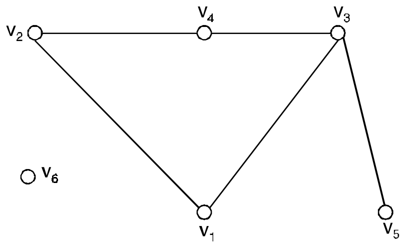


Fig. 3.7(a)

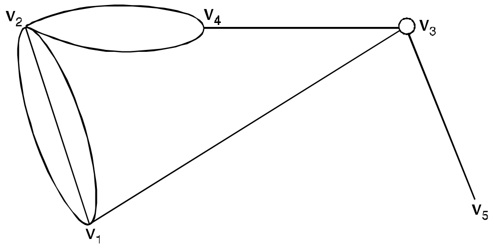


Fig. 3.7(b)

In G as shown in Figure 3.7(a),

$\deg_G(v_2) = 2 = \deg_G(v_4) = \deg_G(v_1)$, $\deg_G(v_3) = 3$ and $\deg_G(v_5) = 1$ and

In H as shown in Figure 3.7(b),

$\deg_H(v_2) = 5$, $\deg_H(v_4) = 3$, $\deg_H(v_3) = 5$, $\deg_H(v_1) = 4$ and $\deg_H(v_5) = 1$.

The degree of a vertex is some times also referred to as its **valency.**

3.5 ISOLATED AND PENDENT VERTICES

3.5.1. Isolated vertex

A vertex having **no incident edge** is called an **isolated vertex.**

In other words, isolated vertices are those with zero degree.

3.5.2. Pendent or end vertex

A vertex of **degree one,** is called a **pendent vertex** or an **end vertex.**

In the above Figure, v_5 is a pendent vertex.

3.5.3. In degree and out degree

In a graph G, the out degree of a vertex v_i of G, denoted by out $\deg_G(v_i)$ or $\deg_G^+(v_i)$, is the number of edges beginning at v_i and the in degree of v_i, denoted by in $\deg_G(\boldsymbol{v_i})$ or $\deg_G^{-1}(\boldsymbol{v_i})$**,** is the number of edges ending at v_i.

The sum of the in degree and out degree of a vertex is called the **total degree** of the vertex. A vertex with zero in degree is called a **source** and a vertex with zero out degree **is called a sink.** Since each edge has an initial vertex and terminal vertex.

3.5.4. The Handshaking Theorem 3.1

If G = (v, E) be an undirected graph with e edges.

Then $\sum_{v \in V} \deg_G(v) = 2e$

i.e., the sum of degrees of the vertices is an undirected graph is even.

Proof : Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end.

Thus the sum of the degrees equal twice the number of edges.

Note : This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

Corollary : In a non directed graph, the total number of odd degree vertices is even.

Proof : Let G = (V, E) a non directed graph.

Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then $\sum_{v_i \in V} \deg_G (v_i) = \sum_{v_i \in U} \deg_G (v_i) + \sum_{v_i \in W} \deg_G (v_i)$

$$\Rightarrow \quad 2e - \sum_{v_i \in U} \deg_G (v_1) = \sum_{v_i \in W} \deg_G (v_1) \qquad ...(1)$$

Now $\sum_{v_i \in W} \deg_G (v_i)$ is also even

Therefore, from (1) $\sum_{v_i \in W} \deg_G (v_i)$ is even

$\therefore$ The no. of odd vertices in G is even.

Problem 3.19. *Find the in degree out degree and of total degree of each vertex of the following graph.*

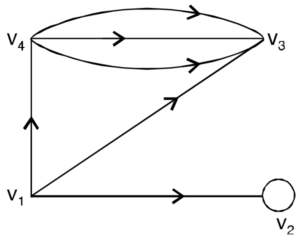


Fig. 3.23

Solution. It is a directed graph

in deg (v_1) = 0,	out deg (v_1) = 3,	total deg (v_1) = 4
in deg (v_2) = 2,	out get (v_2) = 1,	total deg (v_2) = 3
in deg (v_3) = 4,	out deg (v_3) = 0,	total deg (v_3) = 4
in deg (v_4) = 1,	out deg (v_4) = 3,	total deg (v_4) = 4.

Problem 3.20. *State which of the following graphs are simple ?*

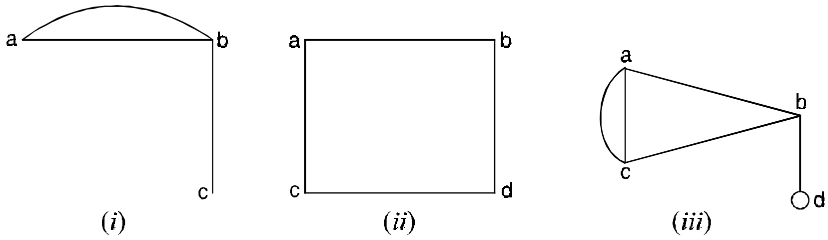


Fig. 3.24

Solution. (*i*) The graph is not a simple graph, since it contains parallel edge between two vertices *a* and *b*.

(*ii*) The graph is a simple graph, it does not contain loop and parallel edge.

(*iii*) The graph is not a simple graph, since it contains parallel edge and a loop.

Problem 3.26. *Consider the following graphs and determine the degree of each vertex :*

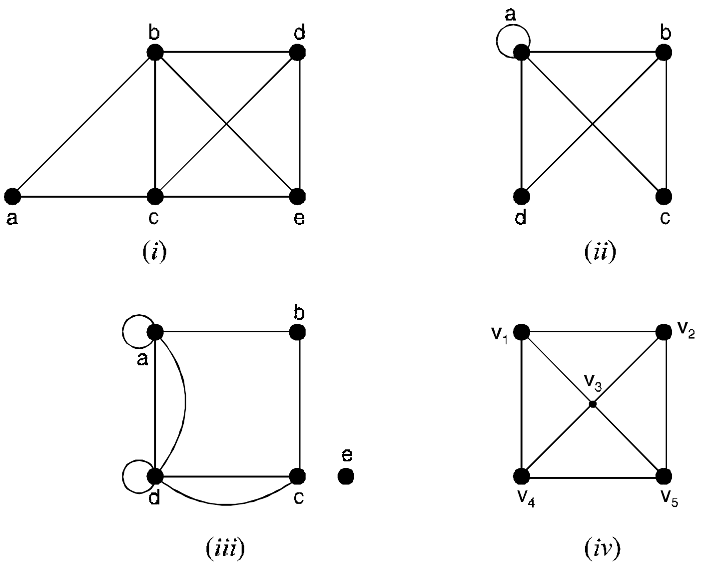


Fig. 3.26

Solution. (*i*) deg (*a*) = 2, deg (*b*) = 4, deg (*c*) = 4, deg (*d*) = 3, deg (*e*) = 3

(*ii*) deg (*a*) = 5, deg (*b*) = 2, deg (*c*) = 3, deg (*d*) = 6, deg (*e*) = 0

(*iii*) deg (*a*) = 5, deg (*b*) = 3, deg (*c*) = 2, deg (*d*) = 2,

(*iv*) Every vertex has degree 4.

Problem 3.27. *Find the in-degree and out-degree of each vertex of the following directed graphs :*

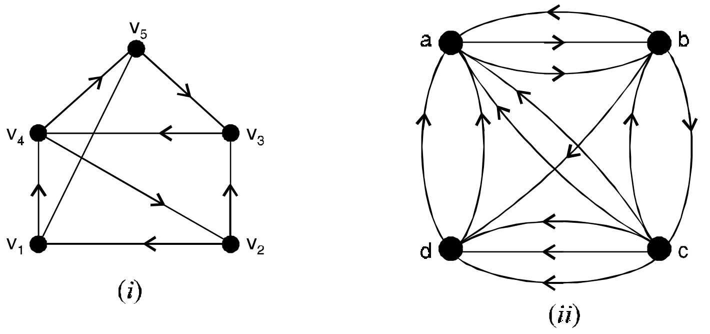


Fig. 3.27

3.6 TYPES OF GRAPHS

Some important types of graph are introduced here.

3.6.1. Null graph

A graph which contains only **isolated node,** is called a null graph.

i.e., the set of edges in a null graph is empty.

Null graph is denoted on n vertices by N_n

N_4 is shown in Figure (3.31), Note that each vertex of a null graph is isolated.

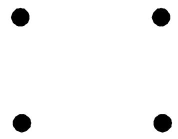


Fig. 3.31

3.6.2. Complete graph

A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex.

i.e., if G contains exactly one edge between each pair of distinct vertices.

A comple graph is usually denoted by $\mathbf{K_n}$. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges.

The graphs K_n for n = 1, 2, 3, 4, 5, 6 are shown in Figure 3.32.

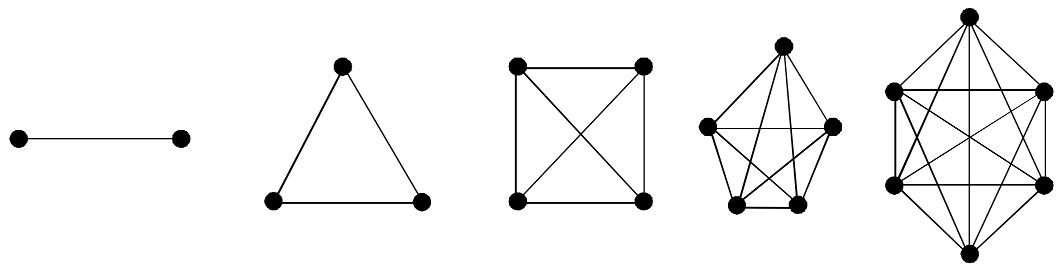


Fig. 3.32

3.6.3. Regular graph

A graph in which all vertices are of **equal degree,** is called a **regular graph.**

If the degree of each vertex is r, then the graph is called a regular **graph of degree** $\boldsymbol{r}$.

Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n - 1$. Also, note that, if G has n vertices and is regular of degree r, then G has $\left(\frac{1}{2}\right) r\,n$ edges.

3.6.4. Cycles

The cycle C_n, $n \geq 3$, consists of n vertices v_1, v_2, ..., v_n and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ..., $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

The cyles c_3, c_4, c_5 and c_6 are shown in Figure 3.33.

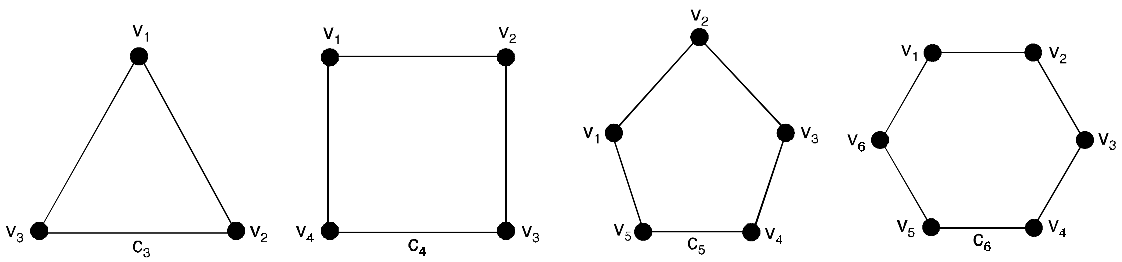


Fig. 3.33. Cycles C_3, C_4, C_5 and C_6.

3.6.5. Wheels

The wheel W_n is obtained when an additional vertex to the cycle c_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in c_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displayed in Figure 3.34.

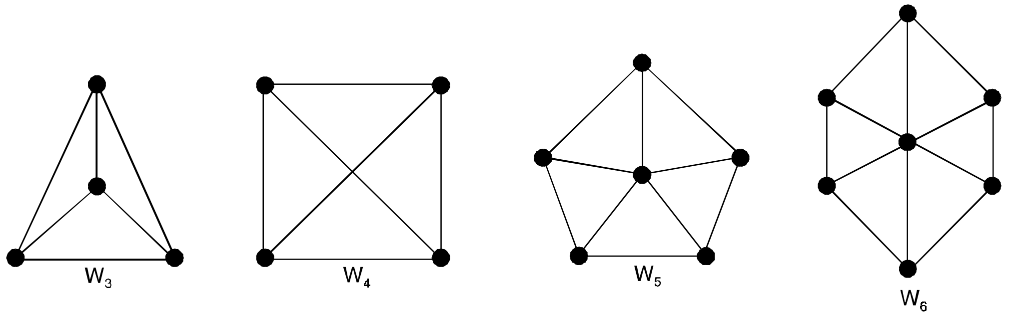


Figure 3.34. The wheels W_3, W_4, W_5 and W_6

3.6.6. Platonic graph

The graph formed by the vertices and edges of the five regular (platonic) solids—The tetrahedron, octahedron, cube, dodecahedron and icosahedron.

The graphs are shown in Figure 3.35.

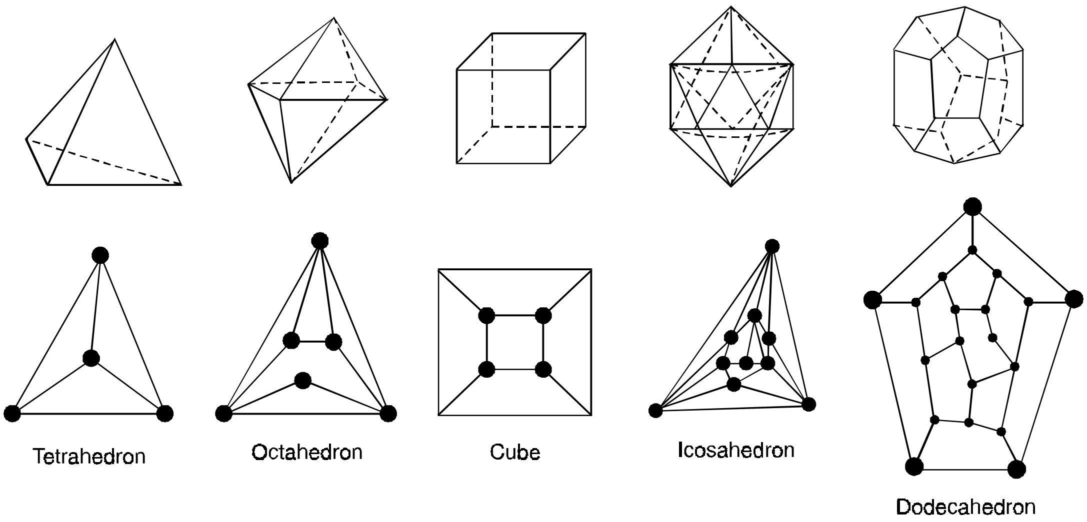


Fig. 3.35.

3.6.7. N-cube

The N-cube denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2, Q_3 are displayed in Figure 3.36. Thus Q_n has 2^n vertices and $n \cdot 2^{n-1}$ edges, and is regular of degree n.

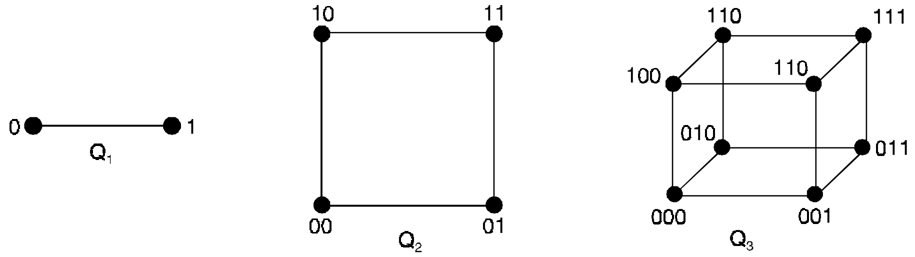
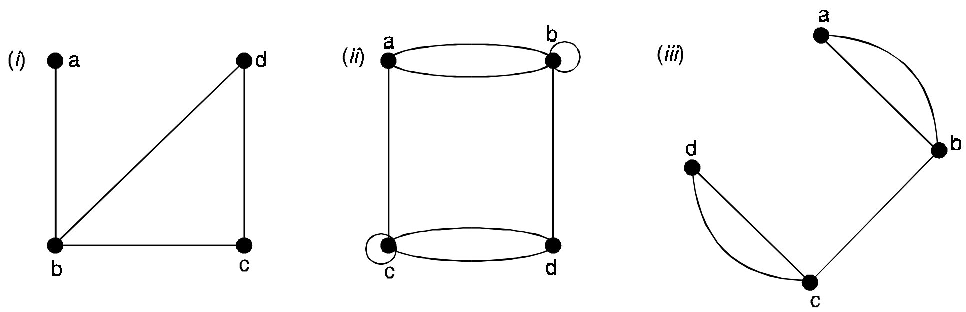


Fig. 3.36. The *n*-cube Q_n for $n = 1, 2, 3$.

Problem 3.34. *Determine whether the graphs shown is a simple graph, a multigraph, a pseudograph.*



Solution. (*i*) Simple graph

(*ii*) Pseudograph

(*iii*) Multigraph.

Problem 3.35. *Consider the following directed graph G : V(G) = {a, b, c, d, e, f, g}*

E(G) = {(a, a), (b, e), (a, e), (e, b), (g, c), (a, e), (d, f), (d, b), (g, g)}.

(*i*) *Identify any loops or parallel edges.*

(*ii*) *Are there any sources in G ?*

(*iii*) *Are there any sinks in G ?*

(*iv*) *Find the subgraph H of G determined by the vertex set V′ = {a, b, c, d}.*

Solution. (*i*) (*a*, *a*) and (*g*, *g*) are loops

(*a*, *a*) and (*a*, *e*) are parallel edges.

(*ii*) No sources

(*iii*) No sinks

(*iv*) V′ = {*a*, *b*, *c*, *d*}

E′ = {(*a*, *a*), (*d*, *b*)}

H = H(V′, E′).

Problem 3.36. *Consider the following graphs, determine the (i) vertex set and (ii) edge set.*

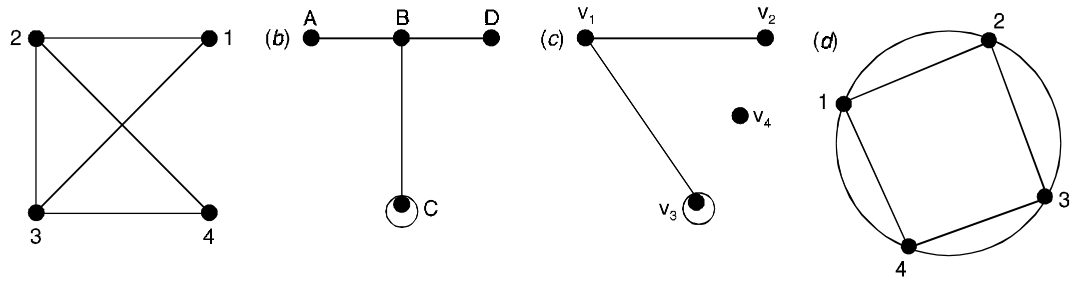


Figure 3.38.

Solution. (*a*) (*i*) Vertex set V = {1, 2, 3, 4},

(*a*) (*ii*) Edge set E = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)}

(*b*) (*i*) Vertex set V = {A, B, C, D}

(*ii*) Edge set E = {(A, B), (B, C), (B, D), (C, C)}

(*c*) (*i*) Vertex set V = $\{v_1, v_2, v_3, v_4\}$

(*ii*) Edge set E = $\{(v_1, v_2), (v_1, v_3), (v_3, v_3)\}$

(*d*) (*i*) Vertex set V = {1, 2, 3, 4}

(*ii*) Edge set E = {(1, 2), (2, 3), (3, 4), (4, 1)}

All edges are double edges.

Fig. 3.39

Problem 3.40. *Draw all six graphs with five vertices and five edges.*

Solution.

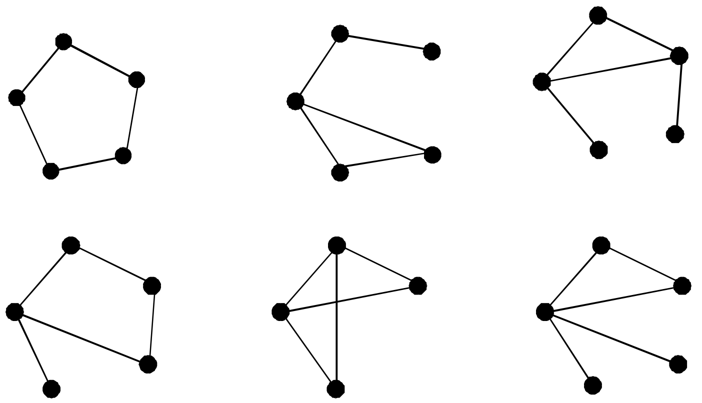


Fig. 3.40

3.7 SUBGRAPHS

A subgraph of G is a graph having all of its vertices and edges in G. If G_1 is a subgraph of G, then G is a super graph of G_1.

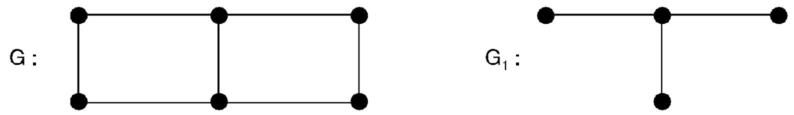


Fig. 3.41. G_1 is a subgraph of G.

In other words. If G and H are two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call H as a subgraph of G or G as a supergraph of H.

3.7.1. Spanning subgraph

A spanning subgraph is a subgraph containing all the vertices of G.

In other words, if $V(H) \subset V(G)$ and $E(H) \subseteq E(G)$ then H is a proper subgraph of G and if V(H) = V(G) then we say that H is a spanning subgraph of G.

A spanning subgraph need not contain all the edges in G.

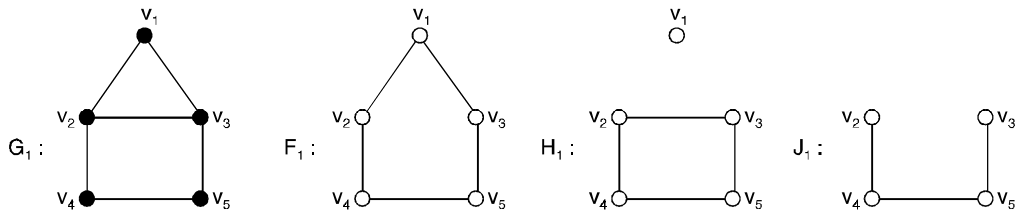


Fig. 3.42.

The graphs F_1 and H_1 of the above Fig. 3.42 are spanning subgraphs of G_1, but J_1 is not a spanning subgraph of G_1.

Since $V_1 \in V(G_1) - V(J_1)$. If E is a set of edges of a graph G, then G – E is a spanning subgraph of G obtained by deleting the edges in E from E(G).

In fact, H is a spanning subgraph of G if and only if H = G – E, where E = E(G) – E(H). If e is an edge of a graph G, then we write $G - e$ instead of $G - \{e\}$. For the graphs G_1, F_1 and H_1 of the Fig. 3.42, we have $F_1 = G_1 - v_2v_3$ and $H_1 = G_1 - \{v_1v_2, v_2v_3\}$.