Delay Differential equations

Chapter One

1.1 Definition: A differential equation is an equation with a function & one or more of its derivatives

1.2 Examples: a) $\frac{dy}{dx} + y^2 = 5x$ "order 1"

(b) $\frac{d^2y}{dx^2} + xy = \sin x$ "order 2"

(c) $\frac{d^3y}{dx^3} + x\frac{dy}{dx} + y = e^x$ "order 3"

(d) $\left(\frac{dy}{dx}\right)^2 + y = 5x^2$ "second degree"

(e) $\frac{d^3y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 5x^2$ "first degree"

1.3 $F(x, y(x), y'(x), ---, y^{(n)}) = 0$
is an ordinary DE of n-th order for the unknown function $y(x)$, where F is given

An important problem for ODE is The initial value problem

$$y'(x) = f(x, y(x))$$
$$y(x_0) = y_0$$

where f is a given real function of two variables x, y & x_0, y_0 are given real numbers.

ODE form a subclass of partial differential equations, which is defined as follows

1.3 Definition: A partial differential equation commonly denoted as PDE is a differential equation containing partial derivatives of the ~~in~~dependent variable (one or more) with more than one independent variable

A partial DE for a function $u(x_1, \dots, x_n)$ is an equation of the form $f(x_1, \dots, x_n; u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \dots) = 0$

$$F(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \dots) = 0$$

The PDE is said to be Linear if f is a Linear function of u & its derivatives. The simple PDE is given by $\frac{\partial u}{\partial x}(x, y) = 0$

In PDEs, we denote the partial derivatives using subscripts, such as:

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

So first order PDE is expressed in the form of

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = 0$$

1.4 Examples: of 2^{nd} order PDE.

(a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (b) $u_{xx} + u_{yy} = 0$

(c) $ux \frac{\partial^2 u}{\partial x^2} + u^2 uy \frac{\partial^2 u}{\partial x \partial y} + uy \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u^3 = 0$

(d) $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$

1.5 Definition: In mathematics, delay differential equations (DDEs) are type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. DDEs are also called time-delay systems, systems with after effect or dead-time, hereditary systems, equations with deviating argument, or differential-difference equation. The simplest constant first order DDE has the form

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_k))$$

Where the time delays (lags) τ_j, $\forall\, j = 1, 2, \ldots, k$ are positive constants. More generally, state dependent delays may depend on the solution, That is $\tau_i = \tau_i(t, y(t))$

1.6 Definition: A delay partial differential equation (DPDE) is an equation, which involves

1. At least Two independent variables
2. An unknown function of the independent variables
3. The behavior of the unknown function at some prior value of the independent variable
4. Partial derivative of the unknown function w.r.t the independent variable.

1.7. Remark: DPDE differs from a PDE in that, it depends not only on the solution at a present stage, but also on the solution at some past stage. If the DE depends on the derivative of the solution at some past stage, Then it is a neutral DPDE. DPDE are also called partial functional DEs as their unknown solutions are used in these equations as functional argument.